A note on SDEs with unbounded drift driven by symmetric stable processes

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Abstract

We prove the existence of weak solutions for stochastic differential equation $dX_t = b(t, X_t)dZ_t + a(t, X_t)dt, X_0 \in \mathbb{R}, t \geq 0$, where Z is a symmetric stable process of index $1 < \alpha < 2$, the coefficients are only measurable and the drift term a can be unbounded. This extends and partially improves the results of N. I. Portenko [9] in the case with $b = 1$ and $a(t, x) = a(x)$. Our approach is based on Krylov type estimates for processes X some variants of which are derived in the note and are of independent interest.

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1 Introduction and preliminary facts

In this note we study the existence of weak solutions of one-dimensional, time-dependent stochastic differential equation

$$dX_t = b(t, X_{t-})dZ_t + a(t, X_t)dt, X_0 = x_0 \in \mathbb{R}, t \geq 0, \tag{1.1}$$

where Z is a symmetric stable process of index $0 < \alpha \leq 2$.

The existence of solutions under assumption of continuity of the coefficients a and b is well known. At the same time, the theory of solutions of equation (1.1) with only *measurable coefficients a and b* is far from being complete. The first result in this direction was obtained by N. V. Krylov [5] in the Brownian motion case, that is when $\alpha = 2$. To be more precise, he proved the existence of weak solutions with measurable coefficients a and b under the assumption that there exist some constants K_1, K_2, and K such that

$$0 < K_1 \leq |b| \leq K_2, \ |a| \leq K. \tag{1.2}$$

To prove the existence of solutions, he used some integral estimates for processes X he was also first to derive. Those estimates are known now as Krylov type estimates and play an important role in the theory of stochastic processes as well as in many applications. In particular, such estimates appear to provide major help to actually construct solutions of corresponding stochastic equations.

Recently, the result of Krylov was generalized [8] for the equation (1.1) with $1 < \alpha < 2$. The proof technique used in [8] was based again on utilizing integral estimates of Krylov type some variants of which were derived in [8] as well. In contrast to [5] where one worked with the corresponding Bellman equation to obtain the estimates, the approach in [8] uses a partial integro-differential equation associated with equation (1.1).

The main goal of this note is to prove the existence of solutions of equation (1.1) with measurable coefficients and *unbounded drift a* so that the condition (1.2) does not hold thus enabling us to construct solutions for new classes of coefficients. In other words, we will prove that equation (1.1) has a solution if $1 < \alpha < 2$ and it holds

$$0 < K_1 \leq |b|^\alpha \leq K_2, a^2 \in L_1, \widehat{a^2} \in L_1, \tag{1.3}$$

where L_1 denotes the space of integrable functions on $\mathbb{R}^2$ and $\widehat{a^2}$ defines the Fourier transform of a^2. To obtain the results, we will need to establish first some Krylov type estimates.

As an example, a function $a^2(t,x) = (tx)^{\beta-1}, x > 0, t > 0$ with $0 < \beta < 1$ will satisfy the assumptions in (1.3). At the same time, it is a (locally) unbounded function thus failing to satisfy the conditions in (1.2).

To our knowledge, there are no existence results known for equation (1.1) with only measurable coefficients and unbounded drift a when $\alpha < 2$, except a particular case of equation

$$dX_t = dZ_t + a(X_t)dt, t \geq 0, X_0 \in \mathbb{R}, \tag{1.4}$$

which was studied by N. I. Portenko [9]. Portenko was able to construct weak solutions for equation (1.4) for the case when $1 < \alpha < 2$ and assuming that there is $p > \frac{1}{\alpha-1}$ so that $a \in L_p(\mathbb{R})$. To prove that, he used the Markov property of the process X and some estimates for its transition density function he was able to derive.

The results of this note, valid for a general equation (1.1) rather than a particular case (1.4), generalize and improve to some extent the results of Portenko as well, at least for $1 < \alpha < 3/2$ since in this case the drift coefficient a is assumed in [9] to be integrable with power $p > 2$ while we require the integrability of a with power $p = 2$.

We start first with some definitions and preliminary facts.

For $\alpha < 2$, Z is a purely jump process and we need to work in a Skorokhod space which we will denote by $\mathbf{D}_{[0,\infty)}(\mathbb{R})$. By definition, $\mathbf{D}_{[0,\infty)}(\mathbb{R})$ is the set of all real-valued functions $z : [0, \infty) \to \mathbb{R}$ with right-continuous trajectories and with finite left limits (also called *cádlág* functions). For simplicity, we shall write $\mathbf{D}$ instead of $\mathbf{D}_{[0,\infty)}(\mathbb{R})$. We will equip $\mathbf{D}$ with the σ-algebra $\mathcal{D}$ generated by the Skorokhod topology. Under $\mathbf{D}^n$ we will understand the n-dimensional Skorokhod space defined as $\mathbf{D}^n = \mathbf{D} \times \ldots \times \mathbf{D}$ with the corresponding σ-algebra $\mathcal{D}^n$ being the direct product of n one-dimensional σ-algebras $\mathcal{D}$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)$. We use the notation $(Z, \mathbb{F})$ to indicate that a process Z is adapted to $\mathbb{F}$. A process $(Z, \mathbb{F})$ is called a symmetric stable process of index $\alpha \in (0, 2]$ if trajectories of Z are cádlag functions and

$$\mathbf{E}\left(\exp\left(i\xi(Z_t - Z_s)\right)|\mathcal{F}_s\right) = \exp\left(-(t-s)c|\xi|^{\alpha}\right)$$

for all $t > s \geq 0$ and $\xi \in \mathbb{R}$, where $c > 0$ is a constant. The function $\psi(\xi) = c|\xi|^{\alpha}$ is called the characteristic exponent of the process Z.

The process Z is a process with independent increments thus a Markov process. Therefore, it can be characterized in terms of Markov processes. For any function $g \in L^{\infty}(\mathbb{R})$ and $t \geq 0$, define the operator

$$(T_t g)(x) := \mathbf{E}\Big(g(x + Z_t)\Big)$$

where $L^{\infty}(\mathbb{R})$ is the Banach space of functions $g : \mathbb{R} \to \mathbb{R}$ with the norm $\|g\|_{\infty} = ess\sup|g(x)|$. For a suitable class of functions $g(x)$, we can define an operator $\mathcal{L}$ often called the infinitesimal generator of the process Z as

$$(\mathcal{L}g)(x) = \lim_{t\downarrow 0} \frac{(T_t g)(x) - g(x)}{t}. \tag{1.5}$$

On another hand, in the case of $\alpha \in (0, 2)$, the process Z as a purely discontinuous Markov process can be described by its Poisson jump measure (jump measure of Z on interval $[0, t]$) defined as

$$N(U \times [0, t]) = \sum_{s \leq t} 1_U(Z_s - Z_{s-}),$$

the number of times before the time t that Z has jumps whose size lies in the set U. The compensating measure of N, say $\hat{N}$, is given by

$$\hat{N}(U,t) = \mathbf{E}N(U \times [0,t]) = t\int_U \frac{1}{|x|^{1+\alpha}}dx.$$

It is known that for $\alpha < 2$

$$(\mathcal{L}g)(x) = \int\limits_{\mathbb{R}\setminus\{0\}} [g(x+z) - g(x) - \mathbf{1}_{\{|z|<1\}}g'(x)z]\frac{c_1}{|z|^{1+\alpha}}dz \qquad (1.6)$$

for any $g \in C_b^2(\mathbb{R})$, where $C_b^2(\mathbb{R})$ is the set of all bounded and twice continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ whose derivatives are also bounded. We shall assume from now on the constant c_1 to be chosen in the way that $\psi(\xi) = 1/2|\xi|^\alpha$. In the case of $\alpha = 2$ the infinitesimal generator of Z is the second derivative operator, that is, $\mathcal{L}g(x) = \frac{1}{2}g''(x)$.

Let $L_p(\mathbb{R}^2), p \geq 1$, define the space of all measurable functions $g : \mathbb{R}^2 \to \mathbb{R}$ such that $(\int_{\mathbb{R}} |g(t,x)|^p dtdx)^{1/p} < \infty$. Then, for any $g \in L_1(\mathbb{R}^2)$, there exists its Fourier transform $\widehat{g}$ defined as

$$\widehat{g}(\tau,\omega) := \int\limits_{\mathbb{R}^2} e^{ix\tau}e^{ix\omega}g(t,x)dtdx, \quad (\tau,\omega) \in \mathbb{R}^2.$$

Moreover, if $\widehat{g} \in L_1(\mathbb{R}^2)$, then also the inverse Fourier transform of the function $\widehat{g}$ exists and

$$g(t,x) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \widehat{g}(\omega)e^{-it\tau}e^{-ix\omega}d\tau d\omega, \quad (t,x) \in \mathbb{R}^2. \qquad (1.7)$$

It is clear that calculating the Fourier transform of a function of two variables can be performed as calculating the single Fourier transform in one variable and then in another, in any order. The next statement is known (see, for example, [2], Proposition 9, ch. 1) and will be used frequently later.

Proposition 1.1 *Let $0 < \alpha \leq 2, g \in C_0^\infty(\mathbb{R}^2)^*$ and assume that $\mathcal{L}g \in L_1(\mathbb{R}^2)$. Then*

$$\widehat{\mathcal{L}g} = -\frac{1}{2}|\omega|^\alpha\widehat{g}.$$

We also introduce the following space of functions associated with the infinitisimal operator $\mathcal{L}$ of a symmetric stable process of index α. For any $u \in C_0^\infty(\mathbb{R}^2)$, define the norm

$$\|u\|_H := \|u\|_{L_2} + \|u_t\|_{L_2} + \|\mathcal{L}u\|_{L_2}. \qquad (1.8)$$

*$C_0^\infty(\mathbb{R}^2)$ defines the class of infinitely differentiable functions with a compact support on $\mathbb{R}^2$

One says that a function $u(t,x) \in L_2(\mathbb{R}^2)$ belongs to the space $H(\mathbb{R}^2)$ if there is a sequence of functions $u^n \in C_0^\infty(\mathbb{R}^2)$ such that $\|u^n\|_H < \infty$ for all $n = 1, 2, ..,$

$$\|u^n - u\|_{L_2} \to 0$$

as $n \to \infty$, and

$$\|u_t^n - u_t^m\|_{L_2} \to 0, \|u^n - u^m\|_{L_2} \to 0, \|\mathcal{L}u^n - \mathcal{L}u^m\|_{L_2} \to 0$$

as $n, m \to \infty$. The space H is then called a *Sobolev space*.

To derive related Krylov type estimates, one needs to address the question of existence of solutions for the following integro-differential equation

$$u_t + |b|^\alpha \mathcal{L}u + au_x - \lambda(1 + |b|^\alpha)u = f \text{ a.e. in } \mathbb{R}^2, \tag{1.9}$$

where $\lambda > 0$ and $f \in L_2(\mathbb{R}^2)$ is a given function.

There is a general approach how to prove the existence of a solution $u \in H(\mathbb{R}^2)$ for an equation similar to (1.9). It is based on the arguments of *the method of continuity* and *the method of a priori estimates* known in theory of classical elliptic and parabolic partial differential equations, see, for example, [6]. The method requires to have established first the corresponding a priori estimates for the norm $\|u\|_H$ in terms of the norm $\|f\|_{L_2}$. Those estimates will be derived in the next section. For some details of the existence proof of a solution u of equation (1.9), we also refer to the Appendix in [8].

2 Some estimates

Let $f \in C_0^\infty(\mathbb{R})$ and assume the conditions (1.3) be satisfied.

First, we are interested in deriving of some *a priori estimates* for the norm $\|u\|_H$ in terms of the norm $\|f\|_{L_2}$.

Lemma 2.1 *Let $u \in C_0^\infty(\mathbb{R}^2)$ be a solution of the equation (1.9). Then, it follows that*

a)

$$\|u\|_H \le M_1 \|f\|_{L_2}; \tag{2.1}$$

b)

$$\sup_{t,x} |u(t,x)| \le M_2 \|f\|_{L_2}, \tag{2.2}$$

where the constants M_1 and M_2 depend on K_1, K_2, α, and a.

Proof. Equation (1.9) implies that

$$\frac{1}{|b|^\alpha}(u_t - \lambda u)^2 + 2(u_t - \lambda u)(\mathcal{L}u - \lambda u) + |b|^\alpha(\mathcal{L}u - \lambda u)^2 \le \frac{2}{|b|^\alpha}(a^2 u_x^2 + f^2).$$

Using the condition (1.3), we obtain

$$\int \frac{1}{K_2}(u_t-\lambda u)^2+2\int (u_t-\lambda u)(\mathcal{L}u-\lambda u)+K_1\int (\mathcal{L}u-\lambda u)^2 \le \frac{2}{K_1}\int (a^2u_x^2+f^2). \tag{2.3}$$

Applying the Plansherel's identity and Proposition 1.1, we have that

$$\int_{\mathbb{R}^2}(u_t-\lambda u)^2 dtdx = \int_{\mathbb{R}^2}|\widehat{u_t-\lambda u}|^2 d\tau dw = \int_{\mathbb{R}^2}|\widehat{u}|^2(\lambda^2+\tau^2)d\tau d\omega, \tag{2.4}$$

and

$$\int_{\mathbb{R}^2}(\mathcal{L}u-\lambda u)^2 dtdx = \int_{\mathbb{R}^2}|\widehat{\mathcal{L}u-\lambda u}|^2 d\tau dw = \int_{\mathbb{R}^2}|\widehat{u}|^2(\lambda+|\omega|^\alpha)^2 d\tau d\omega. \tag{2.5}$$

To estimate the term with $a^2u_x^2$, we notice that, for all $(\tau,\omega)\in\mathbb{R}^2$, it holds

$$|\overline{\widehat{u_x^2}}|(\tau,\omega) \le \int_{\mathbb{R}^2} u_x^2(t,x)dtdx = \|u_x\|_2^2 = \|\widehat{u_x}\|_2^2 = \int_{\mathbb{R}^2}|w|^2|\widehat{u}|^2 d\tau d\omega,$$

where $\bar{z}$ denotes the complex-conjugate of z. Therefore, applying once again the Plansherel's identity yields

$$\begin{aligned}\int_{\mathbb{R}^2} a^2(t,x)u_x^2(t,x)dtdx = Re\int_{\mathbb{R}^2}\widehat{a^2}(\tau,\omega)\overline{\widehat{u_x^2}}(\tau,\omega)d\tau d\omega \le \\ \int_{\mathbb{R}^2}|\widehat{a^2}(\tau,\omega)||\overline{\widehat{u_x^2}}(\tau,\omega)|d\tau d\omega \le \\ \Big(\int_{\mathbb{R}^2}|\widehat{a^2}(\tau,\omega)|d\tau d\omega\Big)\Big(\int_{\mathbb{R}^2}|w|^2|\widehat{u}|^2 d\tau d\omega\Big) = \\ N(a)\Big(\int_{\mathbb{R}^2}|w|^2|\widehat{u}|^2 d\tau d\omega\Big),\end{aligned} \tag{2.6}$$

where the constant $N(a)=\int_{\mathbb{R}^2}|\widehat{a^2}|(\tau,\omega)|d\tau d\omega$ is finite because of the assumption (1.3).

Since $1<\alpha<2$, there will exist $\lambda_0>0$ so that for all $\lambda>\lambda_0$ and all $\omega\in\mathbb{R}$, it holds that

$$|\omega|^2 \le \frac{K_1^2}{4N(a)}(\lambda+|\omega|^\alpha)^2. \tag{2.7}$$

Now, we use identities (2.4)-(2.7) to obtain from (2.3) that

$$\begin{aligned}\frac{1}{K_2}\int_{\mathbb{R}^2}|\widehat{u}|^2(\lambda^2+\tau^2)+2\int_{\mathbb{R}^2}(u_t-\lambda u)(\mathcal{L}u-\lambda u)+\frac{K_1}{2}\int_{\mathbb{R}^2}(\lambda+|\omega|^\alpha)^2|\widehat{u}|^2 \\ \le \frac{2}{K_1}\int_{\mathbb{R}^2}f^2.\end{aligned} \tag{2.8}$$

The last inequality implies

$$\frac{\lambda^2}{K_2}\int_{\mathbb{R}^2}|\widehat{u}|^2+2\int_{\mathbb{R}^2}(u_t-\lambda u)(\mathcal{L}u-\lambda u)+\frac{K_1\lambda^2}{2}\int_{\mathbb{R}^2}|\widehat{u}|^2\leq\frac{2}{K_1}\int_{\mathbb{R}^2}f^2,$$

or

$$(\frac{K_1\lambda^2}{2}+\frac{\lambda^2}{K_2})\|u\|_{L_2}^2+2\int_{\mathbb{R}^2}(u_t-\lambda u)(\mathcal{L}u-\lambda u)\leq\frac{2}{K_1}\int_{\mathbb{R}^2}f^2. \tag{2.9}$$

To estimate the second term on the left-hand side of (2.9), we use again the Plansherel's identity to obtain

$$\int_{\mathbb{R}^2}(u_t-\lambda u)(\mathcal{L}u-\lambda u)=Re[\int_{\mathbb{R}^2}(\lambda+|\omega|^\alpha)(\lambda+i\tau)|\widehat{u}|^2]=$$

$$\int_{\mathbb{R}^2}(\lambda+|\omega|^\alpha)(\lambda)|\widehat{u}|^2\geq\int_{\mathbb{R}^2}\lambda^2|\widehat{u}|^2=\lambda^2\|u\|_{L_2}^2\geq 0.$$

Therefore, we have shown that

$$(\frac{K_1\lambda^2}{2}+\frac{\lambda^2}{K_2}+\lambda^2)\|u\|_{L_2}^2\leq\frac{2}{K_1}\|f\|_{L_2}^2,$$

or

$$\|u\|_{L_2}\leq M\|f\|_{L_2}, \tag{2.10}$$

where the constant M depends on K_1, K_2, α, and $\|a\|_2$.

Obviously,

$$\|\mathcal{L}u\|_{L_2}\leq\|\mathcal{L}u-\lambda u\|_{L_2}+\lambda\|u\|_{L_2},$$

and

$$\|u_t\|_{L_2}\leq\|u_t-\lambda u\|_{L_2}+\lambda\|u\|_{L_2}$$

so that the estimate (2.1) follows then from (2.10), the inequality (2.8) and the established fact that the second term on the left-hand side of (2.3) is non-negative.

To prove (2.2), we use the Fourier inversion formula and the Cauchy-Schwarz inequality to obtain

$$|u(t,x)|^2\leq\Big(\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\widehat{u}|d\tau d\omega\Big)^2$$

$$=\frac{1}{16\pi^4}\Big(\int_{\mathbb{R}^2}|\widehat{u}|\Big(|-2\lambda+i\tau-|\omega|^\alpha|\Big)\Big(|-2\lambda+i\tau-|\omega|^\alpha|\Big)^{-1}d\tau d\omega\Big)^2\leq$$

$$\frac{1}{16\pi^4}I_1I_2,$$

where

$$I_1=\int_{\mathbb{R}^2}|\widehat{u}|^2|-2\lambda+i\tau-|\omega|^\alpha|^2d\tau d\omega$$

and

$$I_2 = \int_{\mathbb{R}^2} |-2\lambda + i\tau - |\omega|^\alpha|^{-2} d\tau d\omega.$$

Since $\alpha \in (1,2)$, it follows that

$$I_2 = \int_{\mathbb{R}^2} \frac{d\tau d\omega}{\tau^2 + (2\lambda + |\omega|^\alpha)^2} = \pi \int_{\mathbb{R}} \frac{d\omega}{2\lambda + |\omega|^\alpha} := M_3 < \infty.$$

The term I_1 can be estimated as

$$I_1 \leq 2 \int_{\mathbb{R}^2} |\widehat{u}|^2 |-\lambda + i\tau|^2 d\tau d\omega + 2 \int_{\mathbb{R}^2} |\widehat{u}|^2 |-\lambda - |\omega|^\alpha|^2 d\tau d\omega =$$

$$2 \int_{\mathbb{R}^2} |\widehat{u_t - \lambda u}|^2 d\tau d\omega + 2 \int_{\mathbb{R}^2} |\widehat{\mathcal{L}u - \lambda u}|^2 d\tau d\omega$$

$$= 2\|u_t - \lambda u\|_{L_2}^2 + 2\|\mathcal{L}u - \lambda u\|_{L_2}^2.$$

Thus, we have shown that

$$|u(t,x)|^2 \leq \frac{M_3}{8\pi^4} \Big(\|u_t - \lambda u\|_{L_2}^2 + \|\mathcal{L}u - \lambda u\|_{L_2}^2 \Big)$$

for all $(t,x) \in \mathbb{R}^2$. The estimate (2.2) follows then from the estimate (2.1). $\Box$

Now, using analytic a priori estimates derived in Lemma 2.1, we are going to prove next corresponding integral estimates of Krylov type for the solutions X of the stochastic equation (1.1).

Fix a nonnegative function $f \in C_0^\infty(\mathbb{R}^2)$. We can assume that the equation (1.9) has a solution $u \in H(\mathbb{R}^2)$.

Let $\psi(t,x) \in C_0^\infty(\mathbb{R}^2)$ be a non-negative function with $\psi(t,x) = 0$ for all (t,x) such that $|t| + |x| \geq 1$ and $\int_{\mathbb{R}^2} \psi(t,x) dt dx = 1$. For $\varepsilon > 0$, we define

$$\psi^{(\varepsilon)}(t,x) = \frac{1}{\varepsilon^2} \psi\Big(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\Big)$$

and let $u^{(\varepsilon)}$ to be the convolution of u with the smooth kernel $\psi^{(\varepsilon)}$:

$$u^{(\varepsilon)}(t,x) = \int_{\mathbb{R}^2} u(s,y) \psi^{(\varepsilon)}(t-s, x-y) ds dy.$$

Clearly, $u^{(\varepsilon)} \in C_0^\infty(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} \psi^{(\varepsilon)}(s,x) ds dx = 1$. Moreover, $u^{(\varepsilon)} \to u$ as $\varepsilon \to 0$ pointwise. We also define

$$u_t^{(\varepsilon)} := \frac{\partial}{\partial t}\Big(u^{(\varepsilon)}\Big) \text{ and } u_x^{(\varepsilon)} := \frac{\partial}{\partial x}\Big(u^{(\varepsilon)}\Big).$$

Now, for $\varepsilon > 0$, let

$$f^{(\varepsilon)} := u_t^{(\varepsilon)} + |b|^\alpha \mathcal{L} u^{(\varepsilon)} + a u_x^{(\varepsilon)} - \lambda(1 + |b|^\alpha) u^{(\varepsilon)}.$$

Because of (1.9), $f^{(\varepsilon)} \to f$ as $\varepsilon \to 0$ pointwise.

For any $m = 1, 2, ...$, define the stopping times

$$\tau_m(X) := \inf\{t \geq 0 : |X_t| > m\}$$

and let $\|f\|_{L_2,m,t}$ be the L_2 norm of the function f on $[0,t] \times [-m,m]$.

The following is a local version of Krylov estimates for processes X satisfying equation (1.1).

Theorem 2.2 *Let X be a solution of the equation (1.1), $1 < \alpha < 2$, and the conditions (1.3) be satisfied. Then, for $t \geq 0, x \in \mathbb{R}$, any measurable function $f : [0,\infty) \times \mathbb{R} \to [0,\infty)$, and $\lambda > \lambda_0$, it holds*

$$\mathbf{E} \int_0^{t \wedge \tau_m(X)} f(t, x + X_s) ds \leq M \|f\|_{L_2,m,t} \tag{2.11}$$

where the constant M depends on K_1, K_2, α, t, m, and a.

Proof. Let $\phi_s = \int_0^s (1 + |b(v, X_v)|^\alpha) dv$. Then, for all $(s,x) \in [0, t \wedge \tau_m(X)) \times \mathbb{R}$, we apply Itó's formula to the function $u^{(\varepsilon)}(s, X_s)e^{-\phi_s}$ to obtain

$$\mathbf{E}u^{(\varepsilon)}(s, X_s)e^{-\lambda\phi_s} - u^{(\varepsilon)}(0,x) =$$

$$\mathbf{E} \int_0^s e^{-\lambda\phi_v} \Big\{ u_t^{(\varepsilon)}(v, X_v) + |b(v, X_v)|^\alpha \mathcal{L}u^{(\varepsilon)}(v, X_v) + a(v, X_v)u_x^{(\varepsilon)}(v, X_v) -$$

$$\lambda(1 + |b|^\alpha(v, X_v))u^{(\varepsilon)}(v, X_v) \Big\} dv = -\mathbf{E} \int_0^s e^{-\lambda\phi_v} f^{(\varepsilon)}(v, X_v) dv.$$

Lemma 2.1 implies that u is a bounded function so that the sequence of functions $u^{(\varepsilon)}, \varepsilon > 0$ is uniformly bounded.

Also, for any fixed m, the sequence $e^{-\lambda\phi_v}|b|^\alpha(v, X_v)f^{(\varepsilon)}(v, X_v)$ converges to $e^{-\lambda\phi_v}|b|^\alpha v, (X_v)f(v, X_v)$ as $\varepsilon \to 0$ a.s. on $[0, t \wedge \tau_m(X)) \times \Omega$ and it holds

$$|a(v, X_v)u_x^{(\varepsilon)}(v, X_v)| \leq K$$

for all $\varepsilon > 0$ for some constant $K > 0$ which is independent of ε.

Thus, using the Lebesgue dominated convergence theorem and letting $\varepsilon \to 0$ in the relation

$$\mathbf{E} \int_0^s e^{-\lambda\phi_v} f^{(\varepsilon)}(v, X_v) dv = u^{(\varepsilon)}(0,x) - \mathbf{E}u^{(\varepsilon)}(s, X_s)e^{-\lambda\phi_s},$$

we obtain

$$\mathbf{E} \int_0^s e^{-\lambda\phi_v} f(v, X_v) dv = u(0,x) - \mathbf{E}u(s, X_s)e^{-\lambda\phi_s}.$$

The above implies

$$\mathbf{E}\int_0^s f(v,X_v)dv \le \sup_{v,x}|u(v,x)| \le M\|f\|_{L_2,m,t}.$$

Using the Fatou's lemma and letting $s \to t$, we obtain

$$\mathbf{E}\int_0^{t\wedge\tau_m(X)} f(v,X_v)dv \le M\|f\|_{L_2,m,t}.$$

The later inequality can be extended to any nonnegative measurable function f by using the standard arguments of a monotone class theorem (see, for example, Theorem 21 in [3]). □

3 Existence of solutions with unbounded drift

As an applications of the integral estimates derived in the previous section, we prove here the existence of solutions for equation (1.1) with only measurable coefficients and (locally) unbounded drift a.

Theorem 3.1 *Let $1 < \alpha < 2$ and conditions (1.3) be satisfied. Then, for any initial value $X_0 \in \mathbb{R}$, there exists a solution of the equation (1.1).*

Proof. The proof of existence of weak solutions using the Krylov type estimates follows similar steps as the original proof of Krylov in [5] for the Brownian motion case.

First, for $n = 1, 2, ...$, we define new functions $a_n = a\wedge n$ and notice that then the coefficients b and a_n satisfy the existence conditions (1.2). Therefore, for a fixed n, there exists a probability space $(\Omega^n, \mathcal{F}^n, \mathbf{P}^n)$ with a filtration $\mathbb{F}^n$ on it and the sequences of processes X^n and Z^n so that Z^n is a symmetric stable process of index α and it holds

$$dX^n_t = b(t, X^n_{t-})dZ^n_t + a_n(t, X^n_t)dt, \quad X^n_0 = x_0 \in \mathbb{R}, \quad t \ge 0. \tag{3.1}$$

Let

$$M^n_t := \int_0^t b(s, X^n_{s-})dZ^n_s \text{ and } Y^n_t := \int_0^t a_n(s, X^n_s)ds$$

so that

$$X^n = x_0 + M^n + Y^n, \quad n \ge 1.$$

As next, we verify that the sequence of processes $H^n := (X^n, M^n, Y^n, Z^n), n \ge 1$, is tight in the sense of weak convergence in $(\mathbb{D}^4, \mathcal{D}^4)$. Due to the well-known Aldous' criterion ([1]), it suffices to show that

$$\lim_{l\to\infty}\limsup_{n\to\infty}\mathbf{P}\Big[\sup_{0\le s\le t}\|H^n_s\| > l\Big] = 0 \tag{3.2}$$

for all $t \geq 0$ and

$$\limsup_{n\to\infty} \mathbf{P}\Big[\|H^n_{t\wedge(\tau^n+\delta_n)} - H^n_{t\wedge\tau^n}\| > \varepsilon\Big] = 0 \tag{3.3}$$

for all $t \geq 0, \varepsilon > 0$, every sequence of $\mathbb{F}$-stopping times τ^n, and every sequence of real numbers δ_n such that $\delta_n \downarrow 0$. Here $\|\cdot\|$ denotes the Euclidean norm of a vector.

First we notice that, as it was shown in [11], there exist constants C_1 and C_2 depending on α only such that for all $t > 0$

$$C_1 \mathbf{E}\int_0^t |\sigma_s|^\alpha ds \leq \sup_{\lambda>0} \lambda^\alpha \mathbf{P}\Big(\sup_{s\leq t} |\int_0^s \sigma_u dZ_u| > \lambda\Big) \leq C_2 \mathbf{E}\int_0^t |\sigma_s|^\alpha ds \tag{3.4}$$

where (σ_t) is a suitable integrand such that the stochastic integral with respect to a symmetric stable process Z exists.

According to the property (3.4), it is enough to verify then that the sequence of processes $([M]^n, Y^n)$ is tight in $(\mathbb{D}^2, \mathcal{D}^2)$ where $[M]^n_t = \int_0^t |b|^\alpha(s, X^n_s)ds.$

Using the integral estimates from Theorem 2.2 and the assumptions (1.3), the conditions (3.2) and (3.3) then follow.

From the tightness of the sequence $\{H^n\}$ we conclude that there exists a subsequence $\{n_k\}, k = 1, 2, \ldots$, a probability space $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbf{P}})$ and the process $\bar H$ on it with values in $(\mathbb{D}^4, \mathcal{D}^4)$ such that H^{n_k} converges weakly (in distribution) to the process $\bar H$ as $k \to \infty$. For simplicity, let $\{n_k\} = \{n\}$.

We use now the well-known embedding principle of Skorokhod (see, e.g. Theorem 2.7 in [4]) to imply the convergence of the sequence $\{H^n\}$ a.s. in the following sense: there exists a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbf{P}})$ and processes $\tilde H = (\tilde X, \tilde M, \tilde Y, \tilde Z)$, $\quad \tilde H^n = (\tilde X^n, \tilde M^n, \tilde Y^n, \tilde Z^n)$, $\quad n = 1, 2, \ldots$, on it such that

1) $\tilde H^n \to \tilde H$ as $n \to \infty$ $\tilde{\mathbf{P}}$-a.s.

2) $\tilde H^n = H^n$ in distribution for all $n = 1, 2, \ldots$.

Using standard measurability arguments ([5], chapter 2), one can prove that the processes $\tilde Z^n$ and $\tilde Z$ are symmetric stable processes of the index α with respect to the augmented filtrations $\tilde{\mathbb{F}}^n$ and $\tilde{\mathbb{F}}$ generated by processes $\tilde H^n$ and $\tilde H$, respectively.

Relying on the above properties 1) and 2), and the equation (3.1), one can show (cf. [5], chapter 2) that

$$\tilde X^n_t = x_0 + \int_0^t b(sX^n_{s-})\tilde Z^n_s + \int_0^t a_n(s, \tilde X^n_s)ds, \quad t \geq 0, \quad \tilde{\mathbf{P}}\text{-a.s.}$$

At the same time, from the properties $1), 2)$ and the quasi-left continuity of the the processes $\tilde X^n$ it follows that

$$\lim_{n\to\infty} \tilde X^n_t = \tilde X_t, \quad t \geq 0, \quad \tilde{\mathbf{P}}\text{-a.s.} \tag{3.5}$$

Hence in order to show that the process $\tilde{X}$ is a solution of the equation (1.1), it is enough to prove that, for all $t \geq 0$,

$$\lim_{n\to\infty} \int_0^t b(s, \tilde{X}_s^n) d\tilde{Z}_s^n = \int_0^t b(s, \tilde{X}_s) d\tilde{Z}s \quad \tilde{\mathbf{P}}\text{- a.s.} \tag{3.6}$$

and

$$\lim_{n\to\infty} \int_0^t a_n(s, \tilde{X}_s^n) ds = \int_0^t a(s, \tilde{X}_s) ds \quad \tilde{\mathbf{P}}\text{- a.s.} \tag{3.7}$$

Now we remark that from the convergence in probability it follows that there is a subsequence for which the convergence with probability one holds. Therefore, to verify (3.6) and (3.7) , it suffices to show that for all $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \tilde{\mathbf{P}}\Big[|\int_0^t b(s, \tilde{X}_s^n) d\tilde{Z}_s^n - \int_0^t b(s, \tilde{X}_s) d\tilde{Z}_s| > \varepsilon \Big] = 0 \tag{3.8}$$

and

$$\lim_{n\to\infty} \tilde{\mathbf{P}}\Big[|\int_0^t a_n(s, \tilde{X}_s^n) ds - \int_0^t a(s, \tilde{X}_s) ds| > \varepsilon \Big] = 0. \tag{3.9}$$

We will also need the following result that can be proven in the same way as Lemma 4.2 in [7].

Lemma 3.2 *Let $\tilde{X}$ be the process as defined above. Then, for any Borel measurable function $f : [0, \infty) \times \mathbb{R} \to [0, \infty)$ and any $t \geq 0$, there exists a sequence $m_k \in (0, \infty), k = 1, 2, \ldots$ such that $m_k \uparrow \infty$ as $k \to \infty$ and it holds*

$$\tilde{\mathbf{E}} \int_0^{t \wedge \tau_{m_k}(\tilde{X})} f(s, \tilde{X}_s) ds \leq M \|f\|_{2, m_k, t},$$

where the constant M depends on K_1, K_2, α, t, m_k, and $\|a\|_2$. Moreover, it holds

$$\tilde{\mathbf{P}}\Big[\tau_m(\tilde{X}^n) < t\Big] \to \tilde{\mathbf{P}}\Big[\tau_m(\tilde{X}) < t\Big] \text{ as } n \to \infty. \tag{3.10}$$

Without loss of generality, we can assume $\{m_k\} = \{m\}$.

Let us prove (3.8). Since b is a bounded function, there is a sequence of Lipshitz continuous functions $b_k, k = 1, 2, \ldots$ so that b_k converges to b in L_2 norm. For a fixed $k_1 \in \mathbb{N}$, we can then estimate

$$\tilde{\mathbf{P}}\Big[|\int_0^t b(s, \tilde{X}_{s-}^n) d\tilde{Z}_s^n - \int_0^t b(s, \tilde{X}_{s-}) d\tilde{Z}_s| > \varepsilon \Big] \leq$$

$$\tilde{\mathbf{P}}\Big[|\int_0^t b_{k_1}(s, \tilde{X}_{s-}^n) d\tilde{Z}_s^n - \int_0^t b_{k_1}(s, \tilde{X}_{s-}) d\tilde{Z}_s| > \frac{\varepsilon}{3} \Big]$$

$$+\tilde{\mathbf{P}}\Big[|\int_0^{t\wedge\tau_m(\tilde{X}^n)} b_{k_1}(s, \tilde{X}_s^n) d\tilde{Z}_s^n - \int_0^{t\wedge\tau_m(\tilde{X}^n)} b(s, \tilde{X}_{s-}^n) d\tilde{Z}_s^n| > \frac{\varepsilon}{3} \Big]$$

$$+\tilde{\mathbf{P}}\Big[|\int_0^{t\wedge\tau_m(\tilde X)} b_{k_1}(s,X_s)d\tilde Z_s - \int_0^{t\wedge\tau_m(\tilde X)} b(\tilde X_{s-})d\tilde Z_s| > \frac{\varepsilon}{3}\Big]$$

$$+\tilde{\mathbf{P}}\Big[\tau_m(\tilde X^n) < t\Big] + \tilde{\mathbf{P}}\Big[\tau_m(\tilde X) < t\Big]$$

$$= J_1(k_1,n) + J_2^m(k_1,n) + J_3^m(k_1) + \tilde{\mathbf{P}}\Big[\tau_m(\tilde X^n) < t\Big] + \tilde{\mathbf{P}}\Big[\tau_m(\tilde X) < t\Big]$$

To show the convergence to 0 of terms $J_2^m(k_1,n)$ and $J_3^m(k_1)$, we use first the Chebyshev's inequality and then Theorem 2.2 and Lemma 3.2, respectively. We obtain

$$J_2^m(k_1,n) = \tilde{\mathbf{P}}\Big[|\int_0^{t\wedge\tau_m(\tilde X^n)} b_{k_1}(s,\tilde X_s^n)d\tilde Z_s^n - \int_0^{t\wedge\tau_m(\tilde X^n)} b(s,\tilde X_{s-}^n)d\tilde Z_s^n| > \frac{\varepsilon}{3}\Big]$$

$$\leq \frac{3}{\epsilon}\tilde{\mathbf{E}}|\int_0^{t\wedge\tau_m(\tilde X^n)} |b_{k_1} - b|^\alpha(s,\tilde X_{s-}^n)ds| \leq \frac{3}{\varepsilon}M\||b_{k_1} - b|^\alpha\|_{2,m,t} \tag{3.11}$$

and

$$J_3^m(k_1) = \tilde{\mathbf{P}}\Big[|\int_0^{t\wedge\tau_m(\tilde X)} b_{k_1}(s,X_s)d\tilde Z_s - \int_0^{t\wedge\tau_m(\tilde X)} b(s,\tilde X_{s-})d\tilde Z_s| > \frac{\varepsilon}{3}\Big]$$

$$\leq \frac{3}{\epsilon}\tilde{\mathbf{E}}|\int_0^{t\wedge\tau_m(\tilde X)} |b_{k_1} - b|^\alpha(\tilde X_{s-})ds| \leq \frac{3}{\varepsilon}M\||b_{k_1} - b|^\alpha\|_{2,m,t} \tag{3.12}$$

where the constant M depends on K_1, K_2, m, t, α, and $\|a\|_2$.

Passing to the limit in (3.11) and (3.12) as $k_1 \to \infty$, we obtain that the right sides of (3.11) and (3.12) converge to 0.

The convergence of the term term $J_1(k_1,n)$ to zero as $k_1 \to \infty$ follows from the Chebyshev's inequality and Skorokhod lemma for stable integrals (see [10], Lemma 2.3).

Finally, because of the property (3.10), the remaining terms $\tilde{\mathbf{P}}\Big[\tau_m(\tilde X^n) < t\Big]$ and $\tilde{\mathbf{P}}\Big[\tau_m(\tilde X) < t\Big]$ can be made arbitrarily small by choosing large enough m for all n due to the fact that the sequence of processes $\tilde X^n$ satisfies the property (3.2). This verifies (3.8). The convergence (3.9) can be verified similarly. We omit the details.

Thus, we have proven the existence of a process $\tilde X$ that solves equation (1.1). □

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