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# THE TANAKA FORMULA FOR SYMMETRIC STABLE PROCESSES WITH INDEX $\alpha$, $0<\alpha<2$*

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**Abstract.** For a symmetric stable process $Z=(Z_t)_{t\geqslant 0}$ of index $0<\alpha<2$, any $a\in\mathbf{R}$, and $\gamma\in(0,2)$ satisfying $\alpha-1<\gamma<\alpha$, we give the explicit form of the Doob–Meyer decomposition of the submartingale $|Z-a|^\gamma=(|Z_t-a|^\gamma)_{t\geqslant 0}$, which consists of $|a|^\gamma$, a stochastic integral with respect to the compensated Poisson random measure associated with $Z$, and a predictable increasing process. If $1<\alpha<2$, then the case $\gamma=\alpha-1$, corresponding to the famous Tanaka formula, is also considered. This extends results of Salminen and Yor [*Tanaka formula for symmetric Lévy processes*, in Séminaire de Probabilités XL, Springer, 2007, pp. 265–285] to general indexes $0<\alpha<2$ using a different approach. Related works are [H. Tanaka, *Z. Wahrsch. Verw. Geb.*, 1 (1963), pp. 251–257], [P. Fitzsimmons and R. K. Getoor, *Ann. Inst. H. Poincaré Probab. Statist.*, 28 (1992), pp. 311–333], [T. Yamada, *Tanaka Formula for Symmetric Stable Processes of Index $\alpha$, $1<\alpha<2$*, manuscript, 1997], and [K. Yamada, *Fractional derivatives of local times of $\alpha$-stable Lévy processes as the limits of occupation time problems*, in Limit Theorems in Probability and Statistics, Vol. II, János Bolyai Math. Soc., 2002, pp. 553–573].

**Key words.** symmetric stable processes, Tanaka's formula, mollifiers, Fourier transform

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**1. Introduction and preliminary facts.** Let $Z$ be a one-dimensional symmetric stable process with index $0<\alpha\leqslant 2$ defined on a complete probability space $(\Omega,\mathscr{F},\mathbf{P})$. If $\alpha=2$, the process $Z$ is a Brownian motion, the only process among the symmetric stable processes with continuous trajectories. If $\alpha<2$, the process $Z$ is a pure jump process. For $\alpha=2$ and any $a\in\mathbf{R}$, the following formula was proved by Tanaka [13] in 1963:

$$|Z_t-a|=|Z_0-a|+\int_0^t \operatorname{sign}(Z_s-a)\,\mathrm{d}Z_s+L^Z(t,a)\quad \mathbf{P}\text{-a.s.},\tag{1.1}$$

where $L^Z(t,a)$ is the local time of the Brownian motion $Z$ at point $a$.

Tanaka's formula turned out to be a very valuable tool in the theory of stochastic processes and their applications.

Yamada [16] and Yamada [15] generalized formula (1.1) to the case of a symmetric stable process $Z$ of index $\alpha\in(1,2)$ in the following form:

$$\begin{aligned}|Z_t-a|^{\alpha-1}&=|Z_0-a|^{\alpha-1}\\&\quad+\int_0^t\int_{\mathbf{R}}\bigl[|Z_{s-}-a+y|^{\alpha-1}-|Z_{s-}-a|^{\alpha-1}\bigr]\,q(\mathrm{d}s,\mathrm{d}y)+C_\alpha L^Z(t,a),\end{aligned}\tag{1.2}$$

where $q(\mathrm{d}s,\mathrm{d}y)$ is the compensated Poisson random measure associated with the Lévy process $Z$, $L^Z(t,a)$ is the local time of $Z$ at the point $a$, and $C_\alpha$ is a constant depending on $\alpha$ only.

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Refining ideas and methods of [15] and [16], in a recent preprint, Tsukada [14] gives a version of formula (1.2) for general *strictly stable* processes $Z$ with parameter $1 < \alpha < 2$. He proves the formula

$$\begin{aligned} F(Z_t - a) &= F(Z_0 - a) \\ &\quad + \int_0^t \int_{\mathbf{R}} \left[ |Z_{s-} - a + y|^{\alpha-1} - |Z_{s-} - a|^{\alpha-1} \right] q(\mathrm{d}s, \mathrm{d}y) + L^Z(t,a), \end{aligned} \tag{1.3}$$

where $F(x) = D(\alpha,\beta)\{1 - \beta\, \mathrm{sign}(x)\}|x|^{\alpha-1}$, $x \in \mathbf{R}$, with strictly positive constant $D(\alpha,\beta)$, and skewness parameter $\beta \in [-1,1]$.

Salminen and Yor [11] also studied the Tanaka formula for symmetric stable processes of index $\alpha \in (1,2)$. One of the main results in [11] is the following formula:

$$|Z_t - a|^\gamma = |Z_0 - a|^\gamma + N_t^{\gamma,a} + C(\alpha,\gamma) \int_0^t |Z_s - a|^{\gamma-\alpha}\, \mathrm{d}s. \tag{1.4}$$

Here $C(\alpha,\gamma)$ is a constant depending on $\alpha$ and $\gamma$, $N_t^{\gamma,a}$ is a martingale, and $\gamma$ is such that $\alpha - 1 < \gamma < \alpha$. As a particular case, they also obtained the formula

$$|Z_t - a|^{\alpha-1} = |Z_0 - a|^{\alpha-1} + N_t^a + C(\alpha) L^Z(t,a),$$

a counterpart of formula (1.2), where $N^a$ is a martingale. However, Salminen and Yor did not identify the form of the processes $N^{\gamma,a}$ and $N^a$.

In the present paper, we focus on the case of *symmetric* stable processes. The main objective is to generalize the results of Salminen and Yor [11] to the case of a *general* symmetric stable process, with *arbitrary* index $\alpha$: $0 < \alpha < 2$. More precisely, we show that, for all $a \in \mathbf{R}$, $t \geqslant 0$, $0 < \alpha < 2$, and $\gamma \in (0,2)$ such that $\alpha - 1 < \gamma < \alpha$, it holds $\mathbf{P}$-a.s. that

$$\begin{aligned} |Z_t - a|^\gamma &= |Z_0 - a|^\gamma + \int_0^t \int_{\mathbf{R}} \left[ |Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma \right] q(\mathrm{d}s, \mathrm{d}y) \\ &\quad + c(\alpha,\gamma) \int_0^t |Z_s - a|^{\gamma-\alpha}\, \mathrm{d}s, \end{aligned} \tag{1.5}$$

where $c(\alpha,\gamma)$ is a constant depending on $\alpha$ and $\gamma$, and the stochastic integral with respect to the compensated Poisson random measure $q$ associated with $Z$ in formula (1.5) is a martingale.

The condition $\alpha - 1 < \gamma < \alpha$ may appear a little strange. However, one should notice that for $0 < \alpha < 1$ we cannot use the exponent $\gamma = \alpha - 1$, which is negative. Moreover, for such $\alpha$, the local time does not exist. On the other hand, even in the case $1 < \alpha < 2$, it should be advantageous to have a representation of $|Z_t - a|^\gamma$ not only for $\gamma = \alpha - 1$ as in the classical case, but also for $\alpha - 1 < \gamma < \alpha$. In particular, we can put $\gamma = 1$ and, in this way, get a representation of $|Z_t - a|$, which seems to be of special interest for some applications. We also note that, in the case $1 < \alpha < 2$, formula (1.5) can be equivalently written by using the local time of the symmetric $\alpha$-stable process applying the occupation times formula to the last term in (1.5).

We stress the fact that, for all $\gamma \in (\alpha - 1, \alpha)$ and $t \geqslant 0$, $|Z_t - a|^\gamma$ is integrable, and the meaning of (1.5) is that the process $|Z - a|^\gamma$ is a nonnegative submartingale with the Doob–Meyer decomposition given by the martingale part

$$|Z_0 - a|^\gamma + \int_0^\cdot \int_{\mathbf{R}} \left[ |Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma \right] q(\mathrm{d}s, \mathrm{d}y)$$

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and the predictable increasing process

$$c(\alpha,\gamma)\int_0^{\cdot} |Z_s - a|^{\gamma-\alpha}\,\mathrm{d}s.$$

We also point out that our methods are different from that used in [11]. Whereas Salminen and Yor [11] employed potential-theoretical methods, our arguments are based on stochastic calculus for Lévy processes and, in particular, on Itô's formula (cf. [8], [14], [15], [16]). Finally, our approach is unified in the arising parameter constellations for $\alpha$ and $\gamma$ and recovers the results of Salminen and Yor [11] for $1 < \alpha < 2$ on the one hand and gives new representation formulas for $0 < \alpha \leqslant 1$ on the other hand.

We first recall some basic properties of a symmetric stable process $Z$. Let $\mathbf{F} = (\mathscr{F}_t)$ be a filtration on $(\Omega, \mathscr{F}, \mathbf{P})$ satisfying the usual conditions. The notation $(Z, \mathbf{F})$ means that $Z$ is adapted to $\mathbf{F}$. We call $(Z, \mathbf{F})$ a symmetric stable process of index $\alpha \in (0, 2]$ if the trajectories of $Z$ are càdlàg functions and

$$\mathbf{E}\big(\exp(i\lambda(Z_t - Z_s)) \,\big|\, \mathscr{F}_s\big) = \exp\big(-(t-s)c|\lambda|^{\alpha}\big) \quad \mathbf{P}\text{-a.s.} \tag{1.6}$$

for all $0 \leqslant s < t$ and $\lambda \in \mathbf{R}$, where $c > 0$ is a constant. The function $\psi(\lambda) = c|\lambda|^{\alpha}$, $\lambda \in \mathbf{R}$, is called the characteristic exponent of the process $Z$.

Because $Z$ is a process with homogeneous and independent increments, it is a homogeneous Markov process. Therefore, it can be characterized in terms of Markov processes. For any bounded measurable real function $g$ and $t \geqslant 0$, we define the operator

$$(P_t g)(z) := \int_{\Omega} g(z + Z_t)\,\mathrm{d}\mathbf{P}, \qquad z \in \mathbf{R}.$$

The family $(P_t)_{t\geqslant 0}$ is called the semigroup of convolution operators associated with $Z$. Now, we can define the so-called infinitesimal generator $\mathscr{A}$ of the process $Z$ as

$$\mathscr{A} g = \lim_{t\downarrow 0} \frac{P_t g - g}{t}, \qquad g \in D(\mathscr{A}), \tag{1.7}$$

where the domain $D(\mathscr{A})$ of $\mathscr{A}$ consists of all bounded measurable real functions $g$ such that the limit in (1.7) exists uniformly. It is known that for $0 < \alpha < 2$ the following proposition holds (cf. [12, section 4.1]).

PROPOSITION 1.1. *Let $C_b^2(\mathbf{R})$ be the space of bounded twice continuously differential functions $g$ such that $g'$ and $g''$ are bounded. Then $C_b^2(\mathbf{R}) \subseteq D(\mathscr{A})$, and*

$$\mathscr{A} g(z) = \int_{\mathbf{R}} \big[g(z+y) - g(z) - \mathbf{1}_{\{|y|\leqslant 1\}} g'(z)\, y\big] \frac{c_1}{|y|^{1+\alpha}}\,\mathrm{d}y, \quad z \in \mathbf{R}, \tag{1.8}$$

*for any $g \in C_b^2(\mathbf{R})$.*

From now on we assume that the constant $c_1$ is chosen so as to have $\psi(\lambda) = |\lambda|^{\alpha}/2$.

On the other hand, in the case of $0 < \alpha < 2$, the process $Z$ as a purely discontinuous Lévy process can be described by its jump measure defined as

$$\mu(A) = \sum_{s\geqslant 0} \mathbf{1}_{\{\Delta Z_s \neq 0\}} \mathbf{1}_A(s, \Delta Z_s), \qquad A \in \mathscr{B}(\mathbf{R}_+ \times \mathbf{R}),$$

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where $\mathscr{B}(\mathbf{R}_+ \times \mathbf{R})$ denotes the $\sigma$-algebra of Borel subsets of $\mathbf{R}_+ \times \mathbf{R}$. It is well known that $\mu$ is a homogeneous Poisson random measure on $\mathbf{R}_+ \times \mathbf{R}$ with intensity measure $m := \mathbf{E}[\mu] = \lambda_+ \otimes \nu$, where $\lambda_+$ is the Lebesgue measure on $\mathbf{R}_+$ and $\nu$ is the Lévy measure of $Z$ given by

$$\nu(U) = \mathbf{E}[\mu([0,1] \times U)] = \int_U \frac{c_1}{|y|^{1+\alpha}}\,\mathrm{d}y, \qquad U \in \mathscr{B}(\mathbf{R}).$$

We also note that the measure $m$ is the predictable compensator of the Poisson random measure $\mu$. The random measure $q := \mu - m$, which is defined on the ring $\mathscr{E} := \{A \in \mathscr{B}(\mathbf{R}_+ \times \mathbf{R})\colon m(A) < +\infty\}$, is called the compensated Poisson random measure associated with the Lévy process $Z$. We refer the reader to Jacod and Shiryaev [6, section II.1] for random measures and integration with respect to them.

For our special Lévy process $Z$, the famous Lévy–Itô decomposition (cf., e.g., [10, Theorem I.42]) yields that, for all $t \geqslant 0$,

$$Z_t = \int_0^t \int_{\{|y|\leqslant 1\}} y\, q(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{\{|y|>1\}} y\, \mu(\mathrm{d}s, \mathrm{d}y).$$

We also recall the Itô formula for $Z$. As usual, let $C^2(\mathbf{R})$ be the space of all twice continuously differentiable functions.

Proposition 1.2 (Itô's formula). *Let $(Z, \mathbf{F})$ be a symmetric stable process of index $\alpha \in (0,2)$ with infinitesimal generator $\mathscr{A}$. Then, for any function $g \in C^2(\mathbf{R})$ and $t \geqslant 0$,*

$$\begin{aligned} g(Z_t) - g(Z_0) &= \int_0^t \int_{\mathbf{R}} \big[g(Z_{s-}+y) - g(Z_{s-})\big]\, q(\mathrm{d}s, \mathrm{d}y) \\ &\quad + \int_0^t \int_{\mathbf{R}} \big[g(Z_s + y) - g(Z_s) - \mathbf{1}_{\{|y|\leqslant 1\}} g'(Z_s)\, y\big] \frac{c_1}{|y|^{1+\alpha}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{1.9}$$

*If, furthermore, $g$ belongs to the space $C_b^2$ of all bounded functions from $C^2(\mathbf{R})$ with bounded first and second derivatives, then*

$$g(Z_t) - g(Z_0) = \int_0^t \int_{\mathbf{R}} \big[g(Z_{s-}+y) - g(Z_s)\big]\, q(\mathrm{d}s, \mathrm{d}y) + \int_0^t \mathscr{A} g(Z_s)\,\mathrm{d}s. \tag{1.10}$$

For the convenience of the reader, a short proof of (1.9) is provided in Appendix A. Formula (1.10) immediately follows from (1.9) and Proposition 1.1.

*Remark* 1.1. (i) Some authors do not distinguish between (1.9) and (1.10), writing the Itô formula in the form of (1.10) also for twice continuously differentiable functions $g$ which are unbounded or have an unbounded first or second derivative. This seems to be incorrect, since, in particular, the infinitesimal generator commonly acts on a Banach space of *bounded* measurable functions $g$ (unless the Banach space is not explicitly specified otherwise).

(ii) For the sake of brevity, for any $g \in C^2(\mathbf{R})$, we introduce the notation

$$\mathscr{G} g(z) := \int_{\mathbf{R}} \big[g(z+y) - g(y) - \mathbf{1}_{\{|y|\leqslant 1\}} g'(z)\, y\big] \frac{c_1}{|y|^{1+\alpha}}\,\mathrm{d}y, \quad z \in \mathbf{R}. \tag{1.11}$$

Obviously, if $g \in C_b^2(\mathbf{R})$, then $\mathscr{G} g = \mathscr{A} g$, so that the linear operator $\mathscr{G}$ on $C^2(\mathbf{R})$ can be considered as an extension of the infinitesimal generator $\mathscr{A}$.

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(iii) In view of the abbreviation (1.11), the Itô formula takes the following form: for any $g \in C^2(\mathbf{R})$ and $t \geqslant 0$,

$$g(Z_t) - g(Z_0) = \int_0^t \int_{\mathbf{R}} \big[g(Z_{s-} + y) - g(Z_{s-})\big]\, q(\mathrm{d}s, \mathrm{d}y) + \int_0^t \mathscr{G} g(Z_{s-})\,\mathrm{d}s.$$

In what follows, as a basic tool we use the Fourier transform, which we are now going to discuss briefly. For $p \geqslant 1$, let $L^p(\mathbf{R})$ denote the space of all measurable functions $g\colon \mathbf{R} \to \mathbf{R}$ such that $(\int_{\mathbf{R}} |g(x)|^p\,\mathrm{d}x)^{1/p} < \infty$. Then, for any $g \in L^1(\mathbf{R})$, there exists its Fourier transform $Fg$ defined as

$$[Fg](\xi) := \int_{\mathbf{R}} e^{ix\xi} g(x)\,\mathrm{d}x, \qquad \xi \in \mathbf{R}.$$

Moreover, if $Fg \in L^1(\mathbf{R})$, then the inverse Fourier transform $F^{-1}$ of the function $Fg$ also exists, and moreover,

$$g(x) = F^{-1}[Fg](x) = \frac{1}{2\pi} \int_{\mathbf{R}} [Fg](\xi) e^{-ix\xi}\,\mathrm{d}\xi, \qquad x \in \mathbf{R}. \tag{1.12}$$

By $S(\mathbf{R})$ we denote the Schwartz space of rapidly decreasing real functions.

PROPOSITION 1.3. *Suppose that* $0 < \alpha \leqslant 2$. *Then the following are true*:
(i) $S(\mathbf{R}) \subseteq D(\mathscr{A})$;
(ii) *for every function* $g \in S(\mathbf{R})$,

$$\mathscr{A} g = -\frac{1}{2} F^{-1}\big[|\xi|^\alpha Fg\big]. \tag{1.13}$$

*Proof.* Since $S(\mathbf{R}) \subseteq C_b^2(\mathbf{R})$ and since $C_b^2(\mathbf{R}) \subseteq D(\mathscr{A})$ by Proposition 1.1, assertion (i) of the proposition is clear. For assertion (ii), we refer the reader to Applebaum [1, Theorem 3.3.3]. Proposition 1.3 is proved.

The following fact, which is used later, is of independent interest. For the case $0 < \beta < \alpha \wedge 1$, it was stated without proof in [8] in a somewhat different and weaker form.

PROPOSITION 1.4. *Let* $Z$ *be a symmetric* $\alpha$*-stable process with* $0 < \alpha < 2$ *and* $-\alpha < \beta < \alpha \wedge 1$. *Then, for any* $t \geqslant 0$ *and* $a \in \mathbf{R}$,

$$\mathbf{E}\biggl[\int_0^t |Z_s - a|^{-\beta}\,\mathrm{d}s\biggr] \leqslant c < +\infty, \tag{1.14}$$

*where* $c$ *is a constant independent of* $a$ *if* $0 \leqslant \beta < \alpha \wedge 1$.

*Proof.* Let $h(s, \cdot\,)$ be the density of $Z_s$. Then

$$\begin{aligned} \mathbf{E}\biggl[\int_0^t |Z_s - a|^{-\beta}\,\mathrm{d}s\biggr] &= \int_0^t \int_{\{|x-a|>1\}} |x-a|^{-\beta} h(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_0^t \int_{\{|x-a|\leqslant 1\}} |x-a|^{-\beta} h(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &=: I_1 + I_2. \end{aligned}$$

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Assume first that $0 \leqslant \beta < \alpha \wedge 1$. The case $\beta = 0$ is trivial, and therefore we assume from now on that $\beta \neq 0$. Since $|x-a|^{-\beta} \leqslant 1$ on $\{|x-a| > 1\}$, it follows that

$$I_1 \leqslant \int_0^t \int_{\{|x-a|>1\}} h(s,x)\,\mathrm{d}x\,\mathrm{d}s \leqslant t < +\infty.$$

To estimate $I_2$, we use the following property satisfied for any $q \geqslant 1$:

$$\|h(s,\cdot)\|_q = K s^{(1-q)/(\alpha q)} \quad \text{for all } s > 0 \tag{1.15}$$

(see, for example, [17]), where $K > 0$ is a constant depending on $q$ and $\alpha$ only.

By Hölder's inequality,

$$\begin{aligned} I_2 &\leqslant \int_0^t \left( \int_{\{|x-a|\leqslant 1\}} |x-a|^{-\beta p}\,\mathrm{d}x \right)^{1/p} \left( \int_{\mathbf{R}} h^q(s,x)\,\mathrm{d}x \right)^{1/q} \mathrm{d}s \\ &= K \left( \int_0^t s^{(1-q)/(\alpha q)}\,\mathrm{d}s \right) \left( \int_{\{|y|\leqslant 1\}} |y|^{-\beta p}\,\mathrm{d}y \right)^{1/p} =: K J_1 J_2, \end{aligned}$$

where $p, q > 1$ are such that $1/p + 1/q = 1$.

If $\alpha \geqslant 1$, then $\beta < 1$. Choosing $1 < p < 1/\beta$, we see that $J_2$ is finite. Now, if $q$ is such that $1/p + 1/q = 1$, then $(1-q)/(\alpha q) = -1/(\alpha p) > -1$, and therefore $J_1$ is also finite.

If $\alpha < 1$, then, in view of the assumption $0 < \beta < \alpha \wedge 1$, we have $\beta < \alpha$. We choose $p$ such that $\beta < 1/p < \alpha$. The first inequality implies that $J_2$ is finite. The second inequality ensures that $q$ chosen such that $1/p + 1/q = 1$ satisfies $(1-q)(\alpha q) = -1/(\alpha p) > -1$, and hence $J_1$ is finite, too.

Therefore, we have shown that (1.14) holds with $c = t + K J_1 J_2$, which is independent of $a$, if $0 < \beta < \alpha \wedge 1$.

It remains to show that for $-\alpha < \beta < 0$ we have $\mathbf{E}\big[\int_0^t |Z_s - a|^{-\beta}\,\mathrm{d}s\big] < +\infty$ or, equivalently, for $0 < \beta < \alpha$

$$\mathbf{E}\left[ \int_0^t |Z_s - a|^{\beta}\,\mathrm{d}s \right] < +\infty.$$

For $0 < \beta \leqslant 1$ we employ the inequality $|a+b|^\beta \leqslant |a|^\beta + |b|^\beta$ (see Appendix B, Proposition B.1(a)) to show that

$$\mathbf{E}\left[ \int_0^t |Z_s - a|^{\beta}\,\mathrm{d}s \right] \leqslant \int_0^t \mathbf{E}|Z_s|^\beta\,\mathrm{d}s + |a|^\beta t.$$

In view of the self-similarity property, following from (1.6), it follows that the distribution of $Z_s$ coincides with that of $s^{1/\alpha} Z_1$. Therefore,

$$\int_0^t \mathbf{E}|Z_s|^\beta\,\mathrm{d}s = \mathbf{E}|Z_1|^\beta \int_0^t s^{\beta/\alpha}\,\mathrm{d}s < +\infty,$$

since it is well known that $\mathbf{E}|Z_1|^\beta < +\infty$ for $\beta < \alpha$.

For $1 < \beta < \alpha$ the proof is similar using the estimate

$$|a+b|^\beta \leqslant 2(|a|^\beta + |b|^\beta)$$

(see Appendix B, Proposition B.1(b)). This completes the proof of Proposition 1.4.

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PROPOSITION 1.5. *Let $Z$ be a symmetric stable process with $0 < \alpha < 2$. Suppose that $0 \leqslant \beta < \alpha \wedge 1$. Then*

(i) *for every $t \geqslant 0$, on the measure space $([0,t] \times \Omega, \mathscr{B}([0,t]) \otimes \mathscr{F}, \lambda_+ \otimes \mathbf{P})$ the family $(|Z_s - a|^{-\beta})_{a \in \mathbf{R}}$ is uniformly integrable*;

(ii) *the function $a \mapsto \mathbf{E}[\int_0^t |Z_s - a|^{-\beta}\,\mathrm{d}s]$ is bounded and continuous.*

*Proof.* To prove (i), we choose $p > 1$ such that $0 < \beta p < \alpha \wedge 1$. Using Proposition 1.4, we find that

$$\mathbf{E}\left[\int_0^t |Z_s - a|^{-\beta p}\,\mathrm{d}s\right] \leqslant c < \infty,$$

and hence the family $(|Z_s - a|^{-\beta})_{a \in \mathbf{R}}$ is bounded in $L^p([0,t] \times \Omega)$. In view of the theorem of de la Vallée Poussin (cf. [9, Theorem II.22]), $(|Z_s - a|^{-\beta})_{a \in \mathbf{R}}$ is uniformly integrable.

To verify (ii), we notice that $a \mapsto \mathbf{E}\big[\int_0^t |Z_s - a|^{-\beta}\,\mathrm{d}s\big]$ is bounded because of Proposition 1.4. The continuity of this function follows from (i) and Lebesgue's theorem for uniformly integrable sequences. Indeed, let $a_n$, $a \in \mathbf{R}$, be such that $a_n \to a$ as $n \to \infty$. Then the sequence $|Z_s - a_n|^{-\beta}$ is uniformly integrable and converges $\lambda_+ \otimes \mathbf{P}$-a.e. to $|Z_s - a|^{-\beta}$. For this we remark that the set $\{|Z_s - a| = 0\}$ has $\lambda_+ \otimes \mathbf{P}$-measure zero. Hence $\mathbf{E}[\int_0^t |Z_s - a_n|^{-\beta}\,\mathrm{d}s]$ converges to $\mathbf{E}\big[\int_0^t |Z_s - a|^{-\beta}\,\mathrm{d}s\big]$. Proposition 1.5 is proved.

**2. Main results.** The following theorem summarizes the main results of the present paper. The proof of this result is postponed to section 3.

THEOREM 2.1. *Let $Z$ be a symmetric stable process of index $0 < \alpha < 2$ defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. Suppose that $\gamma \in (0,2)$.*

(a) *For any $a \in \mathbf{R}$, $t \geqslant 0$, and $\alpha - 1 < \gamma < \alpha$, it holds $\mathbf{P}$-a.s. that*

$$|Z_t - a|^\gamma = |Z_0 - a|^\gamma + M_t^\gamma + c(\alpha,\gamma)\int_0^t |Z_s - a|^{\gamma-\alpha}\,\mathrm{d}s, \tag{2.1}$$

*where $c(\alpha,\gamma)$ is a constant and $M^\gamma$ is the martingale defined as*

$$M_t^\gamma = \int_0^t \int_{\mathbf{R}} \big[|Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma\big]\, q(\mathrm{d}s, \mathrm{d}y). \tag{2.2}$$

*Moreover, if $\alpha - 1 < \gamma < \alpha/2$, then $M^\gamma$ is a square integrable martingale.*

(b) *In the boundary case, that is, if $\alpha > 1$ and $\gamma = \alpha - 1$, we have*

$$|Z_t - a|^{\alpha-1} = |Z_0 - a|^{\alpha-1} + M_t^{\alpha-1} + c(\alpha)L^Z(t,a), \tag{2.3}$$

*where $c(\alpha)$ is a constant, $M^{\alpha-1}$ is the square integrable martingale defined as*

$$M_t^{\alpha-1} = \int_0^t \int_{\mathbf{R}} \big[|Z_{s-} - a + y|^{\alpha-1} - |Z_{s-} - a|^{\alpha-1}\big]\, q(\mathrm{d}s, \mathrm{d}y), \tag{2.4}$$

*and $L^Z(t,a)$ is the local time of the process $Z$ at point $a$.*

COROLLARY 2.1. (a) *The constant $c(\alpha,\gamma)$ in the representation* (2.1) *is given by*

$$c(\alpha,\gamma) = -\frac{\Gamma(\gamma+1)\cos((\gamma+1)\pi/2)}{2\Gamma(\gamma-\alpha+1)\cos((\gamma-\alpha+1)\pi/2)}. \tag{2.5}$$

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(b) *The constant* $c(\alpha)$ *in the representation* (2.3) *is given by*

$$c(\alpha) = -\Gamma(\alpha)\cos\left(\frac{\alpha}{2}\pi\right). \tag{2.6}$$

*Proof.* The calculation of the constants is provided in the proof of Proposition 3.2.

We remark that the constants $c(\alpha)$ and $c(\alpha,\gamma)$ are strictly positive because we always have $\cos((\gamma+1)\pi/2) < 0$ and $\cos(\alpha\pi/2) < 0$ compensating the minus sign in front of the right-hand side of (2.5) and (2.6), respectively.

The next corollary shows that in the case of $1 < \alpha < 2$ the predictable increasing process in the semimartingale decomposition (2.1) of $|Z_t - a|^\gamma$ can be written by using the local time $L^Z(t,y)$ of $Z$.

COROLLARY 2.2. *Suppose that* $1 < \alpha < 2$ *and* $\gamma \in (\alpha-1,\alpha)$. *Then, for any* $a \in \mathbf{R}$ *and* $t \geqslant 0$, *it holds* $\mathbf{P}$*-a.s. that*

$$|Z_t - a|^\gamma = |Z_0 - a|^\gamma + M_t^\gamma + c(\alpha,\gamma)\int_{\mathbf{R}} |y-a|^{\gamma-\alpha} L^Z(t,y)\,\mathrm{d}y, \tag{2.7}$$

*where* $c(\alpha,\gamma)$ *is the constant defined in* (2.5) *and* $M^\gamma$ *is the martingale defined as*

$$M_t^\gamma = \int_0^t \int_{\mathbf{R}} \left[|Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma\right] q(\mathrm{d}s,\mathrm{d}y).$$

*Moreover, if* $\alpha - 1 < \gamma < \alpha/2$, *then* $M^\gamma$ *is a square integrable martingale.*

The representation (2.7) looks closer to the representation (2.3) as given in Theorem 2.1. In comparison with (2.1), the kernel $|y-a|^{\gamma-\alpha}$ has to be integrated with respect to the measure $L^Z(t,y)\,\mathrm{d}y$.

It seems useful that, for $1 < \alpha < 2$ and $\gamma = 1$, we obtain a representation of the modulus $|Z_t - a|$.

COROLLARY 2.3. *Suppose that* $1 < \alpha < 2$. *Then, for any* $a \in \mathbf{R}$ *and* $t \geqslant 0$, *it holds* $\mathbf{P}$*-a.s. that*

$$|Z_t - a| = |Z_0 - a| + M_t + c(\alpha,1)\int_{\mathbf{R}} |y-a|^{1-\alpha} L^Z(t,y)\,\mathrm{d}y, \tag{2.8}$$

*where* $c(\alpha,1)$ *is the constant defined in* (2.5) *and* $M$ *is the martingale defined as*

$$M_t = \int_0^t \int_{\mathbf{R}} \left[|Z_{s-} - a + y| - |Z_{s-} - a|\right] q(\mathrm{d}s,\mathrm{d}y).$$

As a consequence of the following corollary, letting $\gamma$ converge decreasingly to $\alpha - 1$, we see that the representation (2.7) converges to the representation (2.3) from Theorem 2.1.

COROLLARY 2.4. *Let* $1 < \alpha < 2$. *Then, for any* $a \in \mathbf{R}$ *and* $t \geqslant 0$,

(i) $\displaystyle\lim_{\gamma\downarrow\alpha-1} |Z_t - a|^\gamma = |Z_t - a|^{\alpha-1}$,

(ii) $\displaystyle\lim_{\gamma\downarrow\alpha-1} M_t^\gamma = M_t^{\alpha-1}$ $\quad\mathbf{P}$*-a.s.*,

(iii) $\displaystyle\lim_{\gamma\downarrow\alpha-1} c(\alpha,\gamma)\int_{\mathbf{R}} |y-a|^{\gamma-\alpha} L^Z(t,y)\,\mathrm{d}y = c(\alpha)L^Z(t,a)$ $\quad\mathbf{P}$*-a.s.*

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*Proof.* (i) is obvious. A direct proof of (ii) is omitted. However, (ii) follows from (i) and (iii) using Theorem 2.1(a),(b). So it remains to verify (iii). First, we easily see that

$$\lim_{\gamma\downarrow\alpha-1}\left\{-\frac{\Gamma(\gamma+1)\cos((\gamma+1)\pi/2)}{\cos((\gamma-\alpha+1)\pi/2)}\right\} = -\Gamma(\alpha)\cos\left(\frac{\alpha}{2}\pi\right) = c(\alpha).$$

Second, we can conclude that

$$\lim_{\gamma\downarrow\alpha-1}\frac{1}{2\Gamma(\gamma-\alpha+1)}\int_{\mathbf{R}}|y-a|^{\gamma-\alpha}L^Z(t,y)\,\mathrm{d}y = L^Z(t,a) \quad \mathbf{P}\text{-a.s.}$$

This is a consequence of the following facts:

(1) The local time $L^Z(t,y)$ is supported by a compact set $[-N+a,N+a]$ ($N$ depending on $\omega$) $\mathbf{P}$-a.s.;

(2) the finite measures $\mu_N(\alpha,\gamma)$ on $[-N+a,N+a]$ with density

$$\frac{1}{2\Gamma(\gamma-\alpha+1)}|y-a|^{\gamma-\alpha}$$

converge weakly to the Dirac measure $\delta_a$ at $a$ as $\gamma\downarrow\alpha-1$;

(3) the local time $L^Z(t,y)$ is continuous and bounded as a function of $y\in[-N+a, N+a]=[-N(\omega)+a,N(\omega)+a]$.

This proves Corollary 2.4.

## 3. Proof of Theorem 2.1.

**I.** First, we have to verify that the integrals occurring in the formulas (2.1) and (2.4) are well defined and a.s. finite for any $t>0$.

The integral $\int_0^t|Z_s-a|^{\gamma-\alpha}\,\mathrm{d}s$ in (2.1) is well defined and $\mathbf{P}$-a.s. finite since

$$\mathbf{E}\left[\int_0^t|Z_s-a|^{\gamma-\alpha}\,\mathrm{d}s\right] < +\infty.$$

The latter follows directly from Proposition 1.4 applied to $\beta=\alpha-\gamma$.

For the integral in (2.2), we write

$$\int_0^t\int_{\mathbf{R}}\big[|Z_{s-}-a+y|^\gamma-|Z_{s-}-a|^\gamma\big]\,q(\mathrm{d}s,\mathrm{d}y) = M_t^{\gamma,1}+M_t^{\gamma,2},$$

where

$$M_t^{\gamma,1} = \int_0^t\int_{\{|y|\leqslant|Z_s-a|/2\}}\big[|Z_{s-}-a+y|^\gamma-|Z_{s-}-a|^\gamma\big]\,q(\mathrm{d}s,\mathrm{d}y), \tag{3.1}$$

$$M_t^{\gamma,2} = \int_0^t\int_{\{|y|>|Z_s-a|/2\}}\big[|Z_{s-}-a+y|^\gamma-|Z_{s-}-a|^\gamma\big]\,q(\mathrm{d}s,\mathrm{d}y). \tag{3.2}$$

Now, we consider the term

$$J_t^{\gamma,1} := \int_0^t\int_{\{|y|\leqslant|Z_s-a|/2\}}\big[|Z_{s-}-a+y|^\gamma-|Z_{s-}-a|^\gamma\big]^2\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}$$

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and show that $\mathbf{E}[J_t^{\gamma,1}] < +\infty$ for any $t > 0$. With the substitution $y = [Z_s - a]u$, we obtain

$$\begin{aligned}
&\mathbf{E}\bigg[\int_0^t \int_{\{|Z_s-a||u|\leqslant|Z_s-a|/2\}} \big[|(Z_{s-}-a)+(Z_s-a)u|^\gamma \\
&\qquad\qquad - |Z_{s-}-a|^\gamma\big]^2 \frac{c_1\,\mathrm{d}s\,\mathrm{d}u}{|Z_s-a|^\alpha|u|^{1+\alpha}}\bigg] \\
&\quad = \mathbf{E}\bigg[\int_0^t \int_{\{|u|\leqslant 1/2\}} |(Z_{s-}-a)|^{2\gamma}(|1+u|^\gamma-1)^2 \frac{c_1\,\mathrm{d}s\,\mathrm{d}u}{|Z_s-a|^\alpha|u|^{1+\alpha}}\bigg] \\
&\quad = \mathbf{E}\bigg[\int_0^t |Z_s-a|^{2\gamma-\alpha}\,\mathrm{d}s\bigg] \int_{\{|u|\leqslant 1/2\}} (|1+u|^\gamma-1)^2 \frac{c_1\,\mathrm{d}u}{|u|^{1+\alpha}}. \qquad (3.3)
\end{aligned}$$

To estimate the integral in the last expression over the set $\{|u| \leqslant 1/2\}$, we consider the function $f$ defined by $f(u) = |1+u|^\gamma = (1+u)^\gamma$, $u \in (-1/2, 1/2)$, which is differentiable. By the mean value theorem, we get

$$f(u) - f(0) = f'(u_0)u, \qquad 0 < u < \frac{1}{2},$$

with $u_0 \in (0, u)$, and

$$f(u) - f(0) = f'(u_0)u, \qquad -\frac{1}{2} < u < 0,$$

with $u_0 \in (u, 0)$. This means

$$\begin{aligned}
(1+u)^\gamma - 1 &= \gamma(1+u_0)^{\gamma-1}u, \qquad 0 < u < \frac{1}{2}, \\
(1+u)^\gamma - 1 &= \gamma(1+u_0)^{\gamma-1}u, \qquad -\frac{1}{2} < u < 0,
\end{aligned}$$

and therefore

$$|(1+u)^\gamma - 1| \leqslant \gamma c_1(\gamma)|u|,$$

where $c_1(\gamma) = \max\{(3/2)^{\gamma-1}, (1/2)^{\gamma-1}\}$. Consequently, for all $0 < \alpha < 2$,

$$\begin{aligned}
\int_{\{|u|\leqslant 1/2\}} \big||1+u|^\gamma - 1\big|^2 \frac{c_1\,\mathrm{d}u}{|u|^{1+\alpha}} &\leqslant \gamma^2 c_1(\gamma)^2 \int_{\{|u|\leqslant 1/2\}} |u|^{1-\alpha} c_1\,\mathrm{d}u \\
&=: c_2(\alpha,\gamma) < +\infty,
\end{aligned}$$

so that the integral in (3.3) does not exceed

$$c_2(\alpha,\gamma)\mathbf{E}\bigg[\int_0^t |Z_s-a|^{2\gamma-\alpha}\,\mathrm{d}s\bigg].$$

The integral $\mathbf{E}[\int_0^t |Z_s-a|^{2\gamma-\alpha}\,\mathrm{d}s]$ is finite for all $t \geqslant 0$ because of Proposition 1.4. Indeed, with $-\beta = 2\gamma - \alpha$ we have $\beta = \alpha - 2\gamma > \alpha - 2\alpha = -\alpha$. Clearly, $\beta < \alpha$ since $\gamma > 0$. Moreover, $\alpha - 1 \leqslant \gamma < 2\gamma$ which implies $\beta = \alpha - 2\gamma < 1$.

It then follows that the term $\mathbf{E}[J_t^{\gamma,1}]$ is finite for any $t \geqslant 0$ so that $M^{\gamma,1}$ is a well-defined square integrable martingale (cf. [5, section II.3]). Note that the proof also contains the case $\gamma = \alpha - 1$.

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Using the estimates from Proposition B.1 and the same change of variables $y = [Z_s - a]\,u$, it can be shown that the term $\mathbf{E}[J_t^{\gamma,2}]$, where

$$J^{\gamma,2} := \mathbf{E}\biggl[\int_0^t \int_{\{|y|>|Z_s-a|/2\}} \bigl||Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma\bigr| \frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\biggr]$$

is finite for all $t \geqslant 0$. For this we note that

$$\int_{\{|u|>1/2\}} \bigl||1+u|^\gamma - 1\bigr| \frac{c_1\,\mathrm{d}u}{|u|^{1+\alpha}} < +\infty$$

because $\bigl||1+u|^\gamma - 1\bigr|$ behaves like $|u|^\gamma$ as $|u|$ tends to infinity. We omit the details. This implies that $M^{\gamma,2}$ is a well-defined martingale. As a result, the process $M^\gamma$ is a martingale (cf. [5, section II.3]).

On the other hand, by similar arguments, for $\gamma < \alpha/2$, the term

$$\mathbf{E}\biggl[\int_0^t \int_{\{|y|>|Z_s-a|/2\}} \bigl[|Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma\bigr]^2 \frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\biggr]$$

is finite for all $t \geqslant 0$ so that $M^{\gamma,2}$ is then a square integrable martingale. It implies that in this case $M^\gamma$ is a square integrable martingale.

For $1 < \alpha < 2$ and $\gamma = \alpha - 1$, the estimation procedure above works in the same way, and it can be proved in this case that the integral in (2.4) is well defined and is a square integrable martingale.

**II.** Let $V(x) = |x|^\gamma$, $x \in \mathbf{R}$. A naive idea underlying the proof of identity (2.1) is to apply the Itô formula to $V$. However, we cannot do it directly because $V$ is not a twice continuously differentiable function at $x = 0$. To overcome this difficulty, we employ the well-known approach of "smoothing" the function $V$ using mollifiers (cf., e.g., [7, p. 206]).

More precisely, we choose a function $\psi \in C^\infty(\mathbf{R})$ with support in $[-1, 1]$ so that $\psi \geqslant 0$ and $\int_{-1}^1 \psi(y)\,\mathrm{d}y = 1$, and we set $\psi_n(x) := n\psi(nx)$, $x \in \mathbf{R}$, $n = 1, 2, \ldots$ . Then $\psi_n \in C^\infty(\mathbf{R})$ with support in $[-1/n, 1/n]$ and such that $\psi_n \geqslant 0$ and $\int_{\mathbf{R}} \psi_n(y)\,\mathrm{d}y = 1$. Define $V_n = V * \psi_n$, where

$$V * \psi_n(x) = \int_{\mathbf{R}} V(y)\psi_n(x-y)\,\mathrm{d}y, \qquad x \in \mathbf{R},$$

is the convolution of $V$ with the function $\psi_n$. It can be seen that $V_n \in C^\infty(\mathbf{R})$ and $V_n(x) \to |x|^\gamma$ as $n \to \infty$, $x \in \mathbf{R}$. Moreover, $V_n'(x) \to \gamma|x|^{\gamma-1}$ as $n \to \infty$ pointwise for $x \neq 0$.

For any $\varepsilon > 0$, we also set $V_\varepsilon(x) := V(x)e^{-\varepsilon|x|}$, $x \in \mathbf{R}$, and let $V_{\varepsilon,n} := V_\varepsilon * \psi_n$ be the convolution of $V_\varepsilon$ with $\psi_n$.

An application of Itô's formula (1.10) to $V_n$ and $Z - a$ yields

$$\begin{aligned} V_n(Z_t - a) - V_n(Z_0 - a) &= \int_0^t \int \bigl[V_n(Z_{s-} - a + y) - V_n(Z_{s-} - a)\bigr]\, q(\mathrm{d}s, \mathrm{d}y) \\ &+ \int_0^t \int_{\mathbf{R}} \bigl[V_n(Z_{s-} - a + y) - V_n(Z_{s-} - a) - V_n'(Z_{s-} - a)y\mathbf{1}_{\{|y|\leqslant 1\}}\bigr]\, m(\mathrm{d}s, \mathrm{d}y). \end{aligned} \tag{3.4}$$

Let

$$M_t^{n,\gamma} := \int_0^t \int \bigl[V_n(Z_{s-} - a + y) - V_n(Z_{s-} - a)\bigr]\, q(\mathrm{d}s, \mathrm{d}y) \tag{3.5}$$

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and

$$I_t^n := \int_0^t \int_{\mathbf{R}} \big[V_n(Z_{s-}-a+y) - V_n(Z_{s-}-a) - V_n'(Z_{s-}-a)y\mathbf{1}_{\{|y|\leqslant 1\}}\big]\, m(\mathrm{d}s, \mathrm{d}y). \tag{3.6}$$

We first deal with the convergence of the martingales $M^{n,\gamma}$.

PROPOSITION 3.1. *Let $\gamma > 0$ be such that $\alpha - 1 \leqslant \gamma < \alpha$. Then, for all $t > 0$,*

$$\lim_{n\to\infty} M_t^{n,\gamma} = M_t^\gamma \tag{3.7}$$

*in probability and in mean.*

*Proof.* We decompose $M_t^{n,\gamma}$ as $M_t^{n,\gamma} = M_t^{n,\gamma,1} + M_t^{n,\gamma,2}$ with

$$M_t^{n,\gamma,1} := \int_0^t \int_{\mathbf{R}^2} \mathbf{1}_{\{|y|\leqslant |Z_{s-}-a-x|/2\}} \big[|Z_{s-}-a-x+y|^\gamma - |Z_{s-}-a-x|^\gamma\big] \psi_n(x)\, \mathrm{d}x\, q(\mathrm{d}s, \mathrm{d}y)$$

and

$$M_t^{n,\gamma,2} := \int_0^t \int_{\mathbf{R}^2} \mathbf{1}_{\{|y|> |Z_{s-}-a-x|/2\}} \big[|Z_{s-}-a-x+y|^\gamma - |Z_{s-}-a-x|^\gamma\big] \psi_n(x)\, \mathrm{d}x\, q(\mathrm{d}s, \mathrm{d}y).$$

Our aim is to prove that

$$\lim_{n\to\infty} M_t^{n,\gamma,1} = M_t^{\gamma,1} \tag{3.8}$$

in square mean and

$$\lim_{n\to\infty} M_t^{n,\gamma,2} = M_t^{\gamma,2} \tag{3.9}$$

in mean, where the processes $M_t^{\gamma,1}$ and $M_t^{\gamma,2}$ are defined in (3.1) and (3.2) above, respectively. From (3.8) and (3.9), the statement of Proposition 3.1 follows immediately.

To prove (3.8), we set

$$X_n(s,y) := \int_{\mathbf{R}} \mathbf{1}_{\{|y|\leqslant |Z_{s-}-a-x|/2\}} \big[|Z_{s-}-a-x+y|^\gamma - |Z_{s-}-a-x|^\gamma\big] \psi_n(x)\, \mathrm{d}x$$

and

$$X(s,y) := \mathbf{1}_{\{|y|\leqslant |Z_{s-}-a|/2\}} \big[|Z_{s-}-a+y|^\gamma - |Z_{s-}-a|^\gamma\big].$$

We have to show that

$$\mathbf{E}\big[M_t^{n,\gamma,1} - M_t^{\gamma,1}\big]^2 = \mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} (X_n(s,y) - X(s,y))^2\, \frac{c_1\, \mathrm{d}s\, \mathrm{d}y}{|y|^{1+\alpha}}\bigg] \to 0 \tag{3.10}$$

as $n \to \infty$. By Jensen's inequality,

$$\begin{aligned} X_n^2(s,y) &\leqslant \int_{\mathbf{R}} \mathbf{1}_{\{|y|\leqslant |Z_{s-}-a-x|/2\}} \times \big[|Z_{s-}-a-x+y|^\gamma - |Z_{s-}-a-x|^\gamma\big]^2 \psi_n(x)\, \mathrm{d}x \\ &= \int_{-1}^1 Y_n^2(s,y,x)\psi(x)\, \mathrm{d}x =: \overline{Y}_n^2(s,y), \end{aligned} \tag{3.11}$$

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where the function

$$Y_n(s,y,x) := \mathbf{1}_{\{|y|\leqslant|Z_{s-}-a-x/n|/2\}}\left[\left|Z_{s-}-a-\frac{x}{n}+y\right|^{\gamma}-\left|Z_{s-}-a-\frac{x}{n}\right|^{\gamma}\right]$$

is defined on the product space $[0,t]\times\mathbf{R}\times\Omega\times[-1,1]$.

LEMMA 3.1. *Let the probability measure $\rho$ on $[-1,1]$ be defined by*

$$\rho(\mathrm{d}x)=\psi(x)\,\mathrm{d}x.$$

*Then, for any $t\geqslant 0$,*

$$\lim_{n\to\infty}\mathbf{E}\left[\int_0^t\int_{\mathbf{R}}\int_{-1}^{1}Y_n^2(s,y,x)\rho(\mathrm{d}x)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\right]=\mathbf{E}\left[\int_0^t\int_{\mathbf{R}}X^2(s,y)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\right]. \tag{3.12}$$

*Proof.* Using the change of variables $y=(Z_{s-}-a-x/n)u$ and Fubini's theorem, we can calculate

$$\begin{aligned}
&\mathbf{E}\left[\int_0^t\int_{\mathbf{R}}\int_{-1}^{1}Y_n^2(s,y,x)\,\rho(\mathrm{d}x)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\right]\\
&\quad=\mathbf{E}\Bigg[\int_0^t\int_{\mathbf{R}}\int_{-1}^{1}\mathbf{1}_{\{|y|\leqslant|Z_{s-}-a-x/n|/2\}}\\
&\qquad\times\left[\left|Z_{s-}-a-\frac{x}{n}+y\right|^{\gamma}-\left|Z_{s-}-a-\frac{x}{n}\right|^{\gamma}\right]^2\rho(\mathrm{d}x)c_1|y|^{-\alpha-1}\,\mathrm{d}y\,\mathrm{d}s\Bigg]\\
&\quad=\mathbf{E}\Bigg[\int_0^t\int_{\mathbf{R}}\int_{-1}^{1}\mathbf{1}_{\{|u|\leqslant1/2\}}\left|Z_{s-}-a-\frac{x}{n}\right|^{2\gamma}(|u+1|^{\gamma}-1)^2\\
&\qquad\times\left|Z_{s-}-a-\frac{x}{n}\right|^{-\alpha}c_1|u|^{-\alpha-1}\,\mathrm{d}u\,\rho(\mathrm{d}x)\,\mathrm{d}s\Bigg]
\end{aligned}$$

$$=\int_{-1}^{1}\mathbf{E}\left[\int_0^t\left|Z_{s-}-a-\frac{x}{n}\right|^{2\gamma-\alpha}\mathrm{d}s\right]\rho(\mathrm{d}x) \tag{3.13}$$

$$\times\int_{\mathbf{R}}\mathbf{1}_{\{|u|\leqslant1/2\}}(|u+1|^{\gamma}-1)^2c_1|u|^{-\alpha-1}\,\mathrm{d}u. \tag{3.14}$$

For all $|u|\leqslant 1/2$, we have $\big||u+1|^{\gamma}-1\big|\leqslant c_2|u|$, and hence $\big||u+1|^{\gamma}-1\big|^2\leqslant c_3|u|^2$. This yields the convergence of the integral in (3.14).

Now we consider the integral in (3.13). We assume first that $-\beta:=2\gamma-\alpha>0$. Obviously, we obtain

$$0<-\beta=2\gamma-\alpha<\alpha<2.$$

Since $0<-\beta<2$, we can use the estimate (B.2), and for $x\in[-1,1]$ this yields

$$\left|Z_{s-}-a-\frac{x}{n}\right|^{-\beta}\leqslant 2\left(|Z_{s-}-a|^{-\beta}+1\right).$$

Due to Proposition 1.4, the right-hand side is integrable with respect to the measure $\lambda_+\otimes\mathbf{P}\otimes\rho$ since $0<-\beta<\alpha$. An application of Lebesgue's dominated convergence theorem shows that

$$\lim_{n\to\infty}\int_{-1}^{1}\mathbf{E}\left[\int_0^t\left|Z_{s-}-a-\frac{x}{n}\right|^{2\gamma-\alpha}\mathrm{d}s\right]\rho(\mathrm{d}x)=\mathbf{E}\left[\int_0^t|Z_{s-}-a|^{2\gamma-\alpha}\,\mathrm{d}s\right]. \tag{3.15}$$

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This relation is also true in the opposite case $-\beta := 2\gamma-\alpha \leqslant 0$. Indeed, we then have $0 \leqslant \beta = \alpha-2\gamma < \alpha\wedge 1$, and so Proposition 1.5(ii) can be employed. According to this proposition, the function $a \mapsto \mathbf{E}\bigl[\int_0^t |Z_{s-}-a|^{-\beta}\,\mathrm{d}s\bigr]$ is bounded and continuous, and now (3.15) follows by another appeal to Lebesgue's dominated convergence theorem.

As a result of the above calculation, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{E}\biggl[\int_0^t\int_{\mathbf{R}}\int_{-1}^1 Y_n^2(s,y,x)\,\rho(\mathrm{d}x)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\biggr]\\ &\quad= \mathbf{E}\biggl[\int_0^t |Z_{s-}-a|^{2\gamma-\alpha}\,\mathrm{d}s\biggr]\int_{\mathbf{R}}\mathbf{1}_{\{|u|\leqslant 1/2\}}(|u+1|^\gamma-1)^2 c_1|u|^{-\alpha-1}\,\mathrm{d}u. \qquad (3.16)\end{aligned}$$

Using the change of variables $y=(Z_{s-}-a)\,u$, in the same way as above we can show that

$$\begin{aligned}&\mathbf{E}\biggl[\int_0^t\int_{\mathbf{R}} X^2(s,y)\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\biggr]\\ &\quad= \mathbf{E}\biggl[\int_0^t\int_{\mathbf{R}}\mathbf{1}_{\{|y|\leqslant|Z_{s-}-a|/2\}}\bigl[|Z_{s-}-a+y|^\gamma-|Z_{s-}-a|^\gamma\bigr]^2\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\biggr]\\ &\quad= \mathbf{E}\biggl[\int_0^t |Z_{s-}-a|^{2\gamma-\alpha}\,\mathrm{d}s\biggr]\int_{\mathbf{R}}\mathbf{1}_{\{|u|\leqslant 1/2\}}(|u+1|^\gamma-1)^2 c_1|u|^{-\alpha-1}\,\mathrm{d}u. \qquad (3.17)\end{aligned}$$

From (3.16) and (3.17) it follows immediately that the equality (3.12) is valid. This proves Lemma 3.1.

Now we are ready to complete the proof of (3.8). First, we notice that

$$\lim_{n\to\infty} Y_n^2(s,y,x) = X^2(s,y) \quad m\otimes\mathbf{P}\otimes\rho\text{-a.e.}$$

on $[0,t]\times\mathbf{R}\times\Omega\times[-1,1]$. Recall that $m=\lambda_+\otimes\nu$ is the intensity measure of the Poisson random measure $\mu$. In view of Lemma 3.1 and [2, Proposition 21.7] $Y_n^2$ converges to $X^2$ in $L^1(m\otimes\mathbf{P}\otimes\rho)$. This yields that $\overline{Y}_n^2$ introduced in (3.11) also converges to $X^2$, now in $L^1(m\otimes\mathbf{P})$:

$$\begin{aligned}\mathbf{E}\biggl[\int_0^t\int_{\mathbf{R}}|\overline{Y}_n^2-X^2|\,m(\mathrm{d}s,\mathrm{d}y)\biggr] &= \mathbf{E}\biggl[\int_0^t\int_{\mathbf{R}}\biggl|\int_{-1}^1(Y_n^2-X^2)\,\rho(\mathrm{d}x)\biggr|\,m(\mathrm{d}s,\mathrm{d}y)\biggr]\\ &\leqslant \mathbf{E}\biggl[\int_0^t\int_{\mathbf{R}}\int_{-1}^1|Y_n^2-X^2|\,\rho(\mathrm{d}x)\,m(\mathrm{d}s,\mathrm{d}y)\biggr]\to 0\end{aligned}$$

as $n\to\infty$. In particular, according to [2, Proposition 20.4] the sequence $\overline{Y}_n^2$ converges to $X^2$ in the $m\otimes\mathbf{P}$-measure. Now we are in a position to apply [2, Proposition 21.7] to conclude that the sequence $(\overline{Y}_n^2)_{n\geqslant 1}$ is a uniformly integrable subset of $L^1(m\otimes\mathbf{P})$. Since $0\leqslant X_n^2\leqslant\overline{Y}_n^2$ in view of (3.11), we obtain that $(X_n^2)_{n\geqslant 1}$ is also a uniformly integrable subset of $L^1(m\otimes\mathbf{P})$. Note that the sequence $X_n^2$ converges to $X^2$ $m\otimes\mathbf{P}$-a.e. Finally, an application of Lebesgue's convergence theorem to the uniformly integrable sequence $(X_n-X)^2\leqslant 2(X_n^2+X^2)$ yields that (3.10), and hence (3.8) is satisfied.

In order to complete the proof of Proposition 3.1, we have to verify that relation (3.9) is satisfied as well. To this end, we introduce

$$\begin{aligned}X_n(s,y) := &\int_{\mathbf{R}}\mathbf{1}_{\{|y|>|Z_{s-}-a-x|/2\}}\\ &\times\bigl[|Z_{s-}-a-x+y|^\gamma-|Z_{s-}-a-x|^\gamma\bigr]\psi_n(x)\,\mathrm{d}x\end{aligned}$$

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and

$$X(s,y) := \mathbf{1}_{\{|y|>|Z_{s-}-a|/2\}}[|Z_{s-}-a+y|^\gamma - |Z_{s-}-a|^\gamma].$$

We have to show that

$$\mathbf{E}\big|M_t^{n,\gamma,2} - M_t^{\gamma,2}\big| = \mathbf{E}\bigg[\bigg|\int_0^t \int_{\mathbf{R}} [X_n(s,y) - X(s,y)]\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\bigg|\bigg] \to 0 \tag{3.18}$$

as $n\to\infty$. Obviously, we obtain

$$\begin{aligned}|X_n(s,y)| &\leqslant \int_{\mathbf{R}} \mathbf{1}_{\{|y|>|Z_{s-}-a-x|/2\}} \\ &\qquad \times \big||Z_{s-}-a-x+y|^\gamma - |Z_{s-}-a-x|^\gamma\big|\psi_n(x)\,\mathrm{d}x \\ &= \int_{-1}^1 Y_n(s,y,x)\,\rho(\mathrm{d}x) =: \overline{Y}_n(s,y), \end{aligned}\tag{3.19}$$

where $\rho(\mathrm{d}x) = \psi(x)\,\mathrm{d}x$, and the function

$$Y_n(s,y,x) := \mathbf{1}_{\{|y|>|Z_{s-}-a-x/n|/2\}}\bigg|\bigg|Z_{s-}-a-\frac{x}{n}+y\bigg|^\gamma - \bigg|Z_{s-}-a-\frac{x}{n}\bigg|^\gamma\bigg|$$

is defined on the product space $[0,t]\times\mathbf{R}\times\Omega\times[-1,1]$.

The following lemma is a counterpart of Lemma 3.1.

LEMMA 3.2. *Assume the probability measure $\rho$ on $[-1,1]$ is defined by $\rho(\mathrm{d}x) = \psi(x)\,\mathrm{d}x$. Then, for any $t\geqslant 0$,*

$$\begin{aligned}&\lim_{n\to\infty}\mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}}\int_{-1}^1 Y_n(s,y,x)\,\rho(\mathrm{d}x)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\bigg] \\ &\qquad = \mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}} |X(s,y)|\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\bigg].\end{aligned}\tag{3.20}$$

*Proof.* Using the change of variables $y = (Z_{s-}-a-x/n)u$ and Fubini's theorem, we can calculate

$$\begin{aligned}&\mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}}\int_{-1}^1 Y_n(s,y,x)\,\rho(\mathrm{d}x)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\bigg] \\ &= \mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}}\int_{-1}^1 \mathbf{1}_{\{|y|>|Z_{s-}-a-x/n|/2\}} \\ &\qquad \times \bigg|\bigg|Z_{s-}-a-\frac{x}{n}+y\bigg|^\gamma - \bigg|Z_{s-}-a-\frac{x}{n}\bigg|^\gamma\bigg|\rho(\mathrm{d}x)c_1|y|^{-\alpha-1}\,\mathrm{d}y\,\mathrm{d}s\bigg] \\ &= \mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}}\int_{-1}^1 \mathbf{1}_{\{|u|>1/2\}} \\ &\qquad \times \bigg|Z_{s-}-a-\frac{x}{n}\bigg|^\gamma\big||u+1|^\gamma-1\big|\bigg|Z_{s-}-a-\frac{x}{n}\bigg|^{-\alpha} c_1|u|^{-\alpha-1}\,\mathrm{d}u\,\rho(\mathrm{d}x)\,\mathrm{d}s\bigg]\end{aligned}$$

$$= \int_{-1}^1 \mathbf{E}\bigg[\int_0^t \bigg|Z_{s-}-a-\frac{x}{n}\bigg|^{\gamma-\alpha}\mathrm{d}s\bigg]\rho(\mathrm{d}x) \tag{3.21}$$

$$\times \int_{\mathbf{R}} \mathbf{1}_{\{|u|>1/2\}}\big||u+1|^\gamma - 1\big|c_1|u|^{-\alpha-1}\,\mathrm{d}u. \tag{3.22}$$

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Note that $\big||u+1|^\gamma - 1\big|$ behaves like $|u|^\gamma$ as $|u|$ converges to $\infty$, and, since $\gamma < \alpha$, $|u|^{\gamma-\alpha-1}$ is integrable over $\{|y| > 1/2\}$. This yields the convergence of the integral in (3.22).

Now we consider the integral in (3.21). Since $-\beta := \gamma - \alpha < 0$, we then have $0 \leqslant \beta = \alpha - \gamma < \alpha \wedge 1$, and so Proposition 1.5(ii) can be applied. According to this result, the function $a \mapsto \mathbf{E}\big[\int_0^t |Z_{s-} - a|^{-\beta}\, \mathrm{d}s\big]$ is bounded and continuous. An application of Lebesgue's dominated convergence theorem shows that

$$\lim_{n\to\infty} \int_{-1}^1 \mathbf{E}\bigg[\int_0^t \Big|Z_{s-} - a - \frac{x}{n}\Big|^{\gamma-\alpha} \mathrm{d}s\bigg]\,\rho(\mathrm{d}x) = \mathbf{E}\bigg[\int_0^t |Z_{s-} - a|^{\gamma-\alpha}\,\mathrm{d}s\bigg]. \tag{3.23}$$

As a result of the calculation above, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} \int_{-1}^1 Y_n(s,y,x)\,\rho(\mathrm{d}x)\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\bigg] \\ &\qquad = \mathbf{E}\bigg[\int_0^t |Z_{s-} - a|^{\gamma-\alpha}\,\mathrm{d}s\bigg] \int_{\mathbf{R}} \mathbf{1}_{\{|u|>1/2\}}\big||u+1|^\gamma - 1\big| c_1 |u|^{-\alpha-1}\,\mathrm{d}u. \end{aligned} \tag{3.24}$$

Using the change of variables $y = (Z_{s-} - a)\,u$, in the same way as above we can show that

$$\begin{aligned} &\mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} \big|X(s,y)\big|\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}}\bigg] \\ &\qquad = \mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} \mathbf{1}_{\{|y|>|Z_{s-}-a|/2\}}\big||Z_{s-} - a + y|^\gamma - |Z_{s-} - a|^\gamma\big|\,\frac{c_1\,\mathrm{d}s\,\mathrm{d}y}{|y|^{1+\alpha}} \\ &\qquad = \mathbf{E}\bigg[\int_0^t |Z_{s-} - a|^{\gamma-\alpha}\,\mathrm{d}s\bigg] \int_{\mathbf{R}} \mathbf{1}_{\{|u|>1/2\}}\big||u+1|^\gamma - 1\big| c_1 |u|^{-\alpha-1}\,\mathrm{d}u. \end{aligned} \tag{3.25}$$

From (3.24) and (3.25) it follows immediately that the equality (3.20) is valid. This proves Lemma 3.2.

We are now prepared for the final part of the proof of Proposition 3.1, the verification of (3.9). First, we note that

$$\lim_{n\to\infty} Y_n(s,y,x) = |X(s,y)| \quad m\otimes\mathbf{P}\otimes\rho\text{-a.e.}$$

on $[0,t]\times\mathbf{R}\times\Omega\times[-1,1]$. Recall that $m = \lambda_+ \otimes \nu$ is the intensity measure of the Poisson random measure $\mu$. In view of Lemma 3.2 and [2, Proposition 21.7] $Y_n$ converges to $|X|$ in $L^1(m\otimes\mathbf{P}\otimes\rho)$. This shows that $\overline{Y}_n$, as introduced in (3.19), also converges to $|X|$, now in $L^1(m\otimes\mathbf{P})$,

$$\begin{aligned} \mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} \big|\overline{Y}_n - |X|\big|\,m(\mathrm{d}s,\mathrm{d}y)\bigg] &= \mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} \bigg|\int_{-1}^1 (Y_n - |X|)\,\rho(\mathrm{d}x)\bigg|\,m(\mathrm{d}s,\mathrm{d}y)\bigg] \\ &\leqslant \mathbf{E}\bigg[\int_0^t \int_{\mathbf{R}} \int_{-1}^1 \big|Y_n - |X|\big|\,\rho(\mathrm{d}x)\,m(\mathrm{d}s,\mathrm{d}y)\bigg] \to 0 \end{aligned}$$

as $n\to\infty$. In particular, according to [2, Proposition 20.4] the sequence $\overline{Y}_n$ converges to $|X|$ in the $m\otimes\mathbf{P}$-measure. Now we are in a position to apply [2, Proposition 21.7] and to conclude that the sequence $(\overline{Y}_n)_{n\geqslant 1}$ is a uniformly integrable subset of $L^1(m\otimes\mathbf{P})$. Since $0 \leqslant |X_n| \leqslant \overline{Y}_n$ in view of (3.19), we see that $(X_n)_{n\geqslant 1}$ is a uniformly integrable subset of $L^1(m\otimes\mathbf{P})$, too. Note that $X_n$ converges to $X$ $m\otimes\mathbf{P}$-a.e.

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Finally, an application of Lebesgue's convergence theorem for uniformly integrable sequences yields that (3.18), and hence (3.9) is satisfied. The proof of Proposition 3.1 is completed.

**III.** In this part of the proof we are going to investigate the behavior of $\mathscr{A}V_{\varepsilon,n}$ and its limit as $\varepsilon \downarrow 0$.

PROPOSITION 3.2. (a) *For all* $0<\alpha<2$ *and* $\gamma$ *such that* $\alpha-1<\gamma<\alpha$ *and all* $n=1,2,\dots,$ *the function* $V_{\varepsilon,n}$ *belongs to* $D(\mathscr{A})$*, and it holds that*

$$\mathscr{G}V_n(x)=\lim_{\varepsilon\downarrow 0}\mathscr{A}V_{\varepsilon,n}(x)=c(\alpha,\gamma)|x|^{\gamma-\alpha}*\psi_n(x), \qquad x\in\mathbf{R}, \tag{3.26}$$

*where the operator* $\mathscr{G}$ *is defined by* (1.11) *and*

$$c(\alpha,\gamma)=-\frac{\Gamma(\gamma+1)\cos((\gamma+1)\pi/2)}{2\Gamma(\gamma-\alpha+1)\cos((\gamma-\alpha+1)\pi/2)}.$$

(b) *In the boundary case, when* $\gamma=\alpha-1$ (*and thus* $1<\alpha<2$)*, it holds that*

$$\mathscr{G}V_n(x)=\lim_{\varepsilon\downarrow 0}\mathscr{A}V_{\varepsilon,n}(x)=c(\alpha)\psi_n(x), \qquad x\in\mathbf{R}, \tag{3.27}$$

*where* $c(\alpha)=-\Gamma(\alpha)\cos(\alpha\pi/2)$.

*Proof.* First, we notice that the function $V_{\varepsilon,n}$ belongs to the Schwartz space $S(\mathbf{R})$ of rapidly decreasing functions, so that $V_{\varepsilon,n}\in D(\mathscr{A})$ and

$$\mathscr{A}V_{\varepsilon,n}=-\frac{1}{2}F^{-1}[|\xi|^{\alpha}FV_{\varepsilon,n}]$$

(cf. Proposition 1.3).

We calculate $FV_{\varepsilon,n}$. Since $FV_{\varepsilon,n}=F(V_\varepsilon*\psi_n)=(FV_\varepsilon)(F\psi_n)$, we need only calculate $FV_\varepsilon$. Simple integration yields

$$FV_\varepsilon(\xi)=\gamma\Gamma(\gamma)\left[\frac{1}{(\varepsilon+i\xi)^{\gamma+1}}+\frac{1}{(\varepsilon-i\xi)^{\gamma+1}}\right],$$

so that

$$\mathscr{A}V_{\varepsilon,n}=-\frac{1}{2}\gamma\Gamma(\gamma)F^{-1}\left[|\xi|^{\alpha-(1+\gamma)}\left(\frac{|\xi|^{\gamma+1}}{(\varepsilon+i\xi)^{\gamma+1}}+\frac{|\xi|^{\gamma+1}}{(\varepsilon-i\xi)^{\gamma+1}}\right)F\psi_n\right]. \tag{3.28}$$

Note further that for all $\xi\neq 0$ the function

$$\frac{|\xi|^{\gamma+1}}{(\varepsilon+i\xi)^{\gamma+1}}+\frac{|\xi|^{\gamma+1}}{(\varepsilon-i\xi)^{\gamma+1}}$$

of $\varepsilon$ is continuous at $\varepsilon=0$, so that

$$\lim_{\varepsilon\downarrow 0}\left(\frac{|\xi|^{\gamma+1}}{(\varepsilon+i\xi)^{\gamma+1}}+\frac{|\xi|^{\gamma+1}}{(\varepsilon-i\xi)^{\gamma+1}}\right)=2\cos\left(\frac{\gamma+1}{2}\pi\right). \tag{3.29}$$

We now prepare the passage to the limit in (3.28) as $\varepsilon\downarrow 0$. To this end, we first state two lemmas.

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LEMMA 3.3. (a) *For all* $n = 1, 2, \ldots,$

$$\mathscr{G}V_n(x) = \lim_{\varepsilon\downarrow 0} \mathscr{A}V_{\varepsilon,n}(x), \qquad x \in \mathbf{R},$$

*where* $\mathscr{G}$ *is defined by* (1.11).

(b) *Let* $\varepsilon > 0$. *The function* $J_\varepsilon$ *defined by*

$$J_\varepsilon(\xi) := |\xi|^{\alpha-(1+\gamma)}\left(\frac{|\xi|^{\gamma+1}}{(\varepsilon+i\xi)^{\gamma+1}} + \frac{|\xi|^{\gamma+1}}{(\varepsilon-i\xi)^{\gamma+1}}\right)F\psi_n(\xi), \qquad \xi \in \mathbf{R},$$

*is bounded by an integrable function* $J$ *over* $\mathbf{R}$.

*Proof.* (a) It follows from the definition of $V_{\varepsilon,n}$ and the monotone convergence theorem that $V_{\varepsilon,n} \to V_n$ as $\varepsilon \downarrow 0$ pointwise for all $n = 1, 2, \ldots$. Furthermore, we have

$$V'_{\varepsilon,n}(x) = \int_{\mathbf{R}} V(y)\, e^{-\varepsilon|y|}\psi'_n(x-y)\,\mathrm{d}y \to \int_{\mathbf{R}} V(y)\psi'_n(x-y)\,\mathrm{d}y = V'_n(x)$$

as $\varepsilon \downarrow 0$ and, consequently, $V'_{\varepsilon,n} \to V'_n$ pointwise as $\varepsilon \downarrow 0$. As we already know, the functions $\mathscr{G}V_n$ and $\mathscr{A}V_{\varepsilon,n}$ are well defined for $n = 1, 2, \ldots$ and $\varepsilon > 0$.

We are now going to estimate the integrand in

$$\mathscr{A}V_{\varepsilon,n}(x) = \int_{\mathbf{R}} \big[V_{\varepsilon,n}(x+y) - V_{\varepsilon,n}(y) - \mathbf{1}_{\{|y|\leqslant 1\}}V'_{\varepsilon,n}(x)\,y\big]\frac{c_1}{|y|^{1+\alpha}}\,\mathrm{d}y$$

(recall that $V_{\varepsilon,n}$ belongs to $D(\mathscr{A})$ and that formula (1.8) holds) as follows:

$$\begin{aligned} h_{\varepsilon,n}(x,y) &:= \big|V_{\varepsilon,n}(x+y) - V_{\varepsilon,n}(x) - \mathbf{1}_{\{|y|\leqslant 1\}}V'_{\varepsilon,n}(x)y\big| \\ &= \left|\int_{\mathbf{R}} V_\varepsilon(z)[\psi_n(x+y-z) - \psi_n(x-z) - \mathbf{1}_{\{|y|\leqslant 1\}}y\psi'_n(x-z)]\,\mathrm{d}z\right| \\ &= \left|\int_{\mathbf{R}} V_\varepsilon(x-z)[\psi_n(z+y) - \psi_n(z) - \mathbf{1}_{\{|y|\leqslant 1\}}y\psi'_n(z)]\,\mathrm{d}z\right| \\ &\leqslant \int_{\mathbf{R}} V(x-z)\big|\psi_n(z+y) - \psi_n(z) - \mathbf{1}_{\{|y|\leqslant 1\}}y\psi'_n(z)\big|\,\mathrm{d}z. \end{aligned}$$

Using the mean value theorem, on the set $\{0 \leqslant y \leqslant 1\}$ we get

$$\begin{aligned} h_{\varepsilon,n}(x,y) &\leqslant \int_{\mathbf{R}} V(x-z)|\psi''_n(\theta_{z,y})y^2|\,\mathrm{d}z = \int_{1/n-1}^{1/n} V(x-z)|\psi''_n(\theta_{z,y})y^2|\,\mathrm{d}z \\ &\leqslant cy^2\int_{1/n-1}^{1/n} V(x-z)\,\mathrm{d}z, \end{aligned}$$

where $\theta_{z,y} \in (z, z+y)$, the next-to-last equality is satisfied, since $\psi''_n(\theta_{z,y}) = 0$ for $z \notin [1/n-1, 1/n]$, and $\psi''_n(\theta_{z,y})$ is bounded by a constant $c$. The right-hand side is integrable with respect to $y$ and the measure $\nu(\mathrm{d}y) = c_1|y|^{-(1+\alpha)}\,\mathrm{d}y$ on $\{0 \leqslant y \leqslant 1\}$. Similarly, it can be shown that, for fixed $n$ and $x$, $h_{\varepsilon,n}(x,y)$ is bounded by an integrable function independent of $\varepsilon$ with respect to $\nu(\mathrm{d}y) = c_1|y|^{-(1+\alpha)}\,\mathrm{d}y$ on $\{-1 \leqslant y < 0\}$.

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Now we estimate $h_{\varepsilon,n}(x,y)$ on the set $\{y>1\}$. Using the main theorem of calculus, we obtain

$$\begin{aligned} h_{\varepsilon,n}(x,y) &= |V_{\varepsilon,n}(x+y)-V_{\varepsilon,n}(x)| = \left|\int_{\mathbf{R}} V_\varepsilon(x-z)(\psi_n(z+y)-\psi_n(z))\,\mathrm{d}z\right| \\ &= \left|\int_{\mathbf{R}} V_\varepsilon(x-z)\int_z^{z+y}\psi_n'(u)\,\mathrm{d}u\,\mathrm{d}z\right| \leqslant \int_{\mathbf{R}} V_\varepsilon(x-z)\int_z^{z+y}|\psi_n'(u)|\,\mathrm{d}u\,\mathrm{d}z \\ &\leqslant \int_{\mathbf{R}} V(x-z)\int_z^{z+y} c(1+|u|)^{-4}\,\mathrm{d}u\,\mathrm{d}z, \end{aligned}$$

where we have used that $\psi_n' \in S(\mathbf{R})$, and therefore $|\psi_n'(u)| \leqslant c(1+|u|)^{-4}$ for all $u\in\mathbf{R}$ and some $c>0$, and that $V_\varepsilon \leqslant V$. Note that a primitive of $(1+|u|)^{-4}$ is $-\operatorname{sign}(u)(1+|u|)^{-3}/3$. Calculating and estimating the inner integral yields

$$h_{\varepsilon,n}(x,y) \leqslant \frac{1}{3}c\left[\int_{\mathbf{R}}|x-z|^\gamma(1+|z+y|)^{-3}\,\mathrm{d}z + \int_{\mathbf{R}}|x-z|^\gamma(1+|z|)^{-3}\,\mathrm{d}z\right].$$

The right-hand side is integrable with respect to the measure $\nu(\mathrm{d}y)=c_1|y|^{-(1+\alpha)}\,\mathrm{d}y$ on $\{y>1\}$. Indeed, the second integral is, for fixed $x$, a finite constant which is integrable with respect to the Lévy measure $\nu$ on $\{y>1\}$. Repeated application of the inequality (B.2) yields

$$|x-(z-y)|^\gamma \leqslant 2(|x|^\gamma+|z-y|^\gamma) \leqslant 2|x|^\gamma + 4(|z|^\gamma+|y|^\gamma),$$

and the first integral can be estimated as follows:

$$\begin{aligned} &\int_{\mathbf{R}}|x-z|^\gamma(1+|z+y|)^{-3}\,\mathrm{d}z = \int_{\mathbf{R}}|x-(z-y)|^\gamma(1+|z|)^{-3}\,\mathrm{d}z \\ &\leqslant 2|x|^\gamma\int_{\mathbf{R}}(1+|z|)^{-3}\mathrm{d}z + 4\int_{\mathbf{R}}|z|^\gamma(1+|z|)^{-3}\,\mathrm{d}z + 4|y|^\gamma\int_{\mathbf{R}}(1+|z|)^{-3}\,\mathrm{d}z. \end{aligned}$$

The right-hand side, as a function of $y$, is integrable with respect to the Lévy measure $\nu$ over $\{y>1\}$, and the claim follows.

Similarly, it can be easily shown that, for fixed $n$ and $x$, $h_{\varepsilon,n}(x,y)$ is bounded by an integrable function independent of $\varepsilon$ with respect to the Lévy measure $\nu(\mathrm{d}y) = c_1|y|^{-(1+\alpha)}\,\mathrm{d}y$ on $\{y<-1\}$.

In other words, the integrand $h_{\varepsilon,n}(x,y)$ can be estimated, for all $n$ and $x$, by a function independent of $\varepsilon$ which is integrable with respect to the Lévy measure $\nu(\mathrm{d}y)=c_1|y|^{-(1+\alpha)}\,\mathrm{d}y$ on the set $\{|y|>1\}$.

Splitting the integral over $\mathbf{R}$ into two parts, one over the set $\{|y|\leqslant 1\}$ and the other over the set $\{|y|>1\}$, and using Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{\varepsilon\downarrow 0}\mathscr{A}V_n^\varepsilon(x) = \mathscr{G}V_n(x) \quad \text{for all } x\in\mathbf{R}. \tag{3.30}$$

(b) Since $|1/(u\pm i)|\leqslant 1$, $u\in\mathbf{R}$, we estimate

$$\left|\frac{|\xi|^{\gamma+1}}{(\varepsilon+i\xi)^{\gamma+1}} + \frac{|\xi|^{\gamma+1}}{(\varepsilon-i\xi)^{\gamma+1}}\right| \leqslant 2,$$

so that

$$|J_\varepsilon(\xi)| \leqslant J(\xi) := 2|\xi|^{\alpha-(1+\gamma)}|F\psi_n(\xi)|.$$

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Taking into account that $-1 < \alpha - (1+\gamma) < 0$ and that $F\psi_n$ is bounded, we conclude that $J$ is integrable in every bounded neighborhood of zero. We also note that $\psi_n \in S(\mathbf{R})$ and $F\psi_n \in S(\mathbf{R})$. Consequently, for any $n$, on $\{|\xi| \geqslant 1\}$

$$|F\psi_n(\xi)| \leqslant c|\xi|^{-2},$$

and therefore

$$J \leqslant 2c|\xi|^{\alpha-3-\gamma},$$

which is integrable on $\{|\xi| \geqslant 1\}$. This proves Lemma 3.3.

LEMMA 3.4. *It holds that*

$$|x|^{\gamma-\alpha} * \psi_n = c_1(\alpha,\gamma) F^{-1}[|\xi|^{\alpha-(1+\gamma)} F\psi_n],$$

*where* $c_1(\alpha,\gamma) = 2\Gamma(\gamma-\alpha+1)\cos((\gamma-\alpha+1)\pi/2)$.

*Proof*. For fixed $N \geqslant 1$, the function $(\mathbf{1}_{[-N,\,N]}(x)|x|^{\gamma-\alpha}) * \psi_n$ is integrable over $\mathbf{R}$ so that its Fourier transform is well defined, and using the convolution formula, we obtain

$$\begin{aligned} F\big[(\mathbf{1}_{[-N,N]}(x)|x|^{\gamma-\alpha}) * \psi_n\big](\xi) &= F\big[(\mathbf{1}_{[-N,N]}(x)|x|^{\gamma-\alpha})\big](\xi) F\psi_n(\xi) \\ &= \left(\int_{-N}^{N} e^{ix\xi}|x|^{\gamma-\alpha}\,\mathrm{d}x\right) F\psi_n(\xi), \qquad \xi \in \mathbf{R}. \end{aligned} \tag{3.31}$$

Denoting

$$c(N,\xi,\gamma,\alpha) := \int_{-N}^{N} e^{ix\xi}|x|^{\gamma-\alpha}\,\mathrm{d}x, \tag{3.32}$$

we easily see that $|c(N,\xi,\gamma,\alpha)| \leqslant 2N^{\gamma-\alpha+1}$, and, in view of $F\psi_n \in S(\mathbf{R})$, the right-hand side of (3.31) is integrable. Therefore, we can apply the inversion formula for the Fourier transform to conclude that

$$(\mathbf{1}_{[-N,\,N]}(x)|x|^{\gamma-\alpha}) * \psi_n(x) = F^{-1}[c(N,\cdot,\gamma,\alpha)F\psi_n](x), \qquad x \in \mathbf{R}. \tag{3.33}$$

For $\xi \neq 0$, we now calculate

$$\begin{aligned} \int_{-N}^{N} e^{ix\xi}|x|^{\gamma-\alpha}\,\mathrm{d}x &= \int_{-N}^{N} e^{ix|\xi|}|x|^{\gamma-\alpha}\,\mathrm{d}x = \int_{-N|\xi|}^{N|\xi|} e^{iz}\left|\frac{z}{\xi}\right|^{\gamma-\alpha}\frac{1}{|\xi|}\,\mathrm{d}z \\ &= |\xi|^{\alpha-(1+\gamma)}\int_{-N|\xi|}^{N|\xi|}|z|^{\gamma-\alpha}e^{iz}\,\mathrm{d}z = |\xi|^{\alpha-(1+\gamma)}\left(2\int_0^{N|\xi|} z^{\gamma-\alpha}\cos z\,\mathrm{d}z\right). \end{aligned} \tag{3.34}$$

Using Fourier transform tables (cf., e.g., [3, Chap. 13.3, formula 5]), we obtain that the improper integral $\int_0^\infty z^{\gamma-\alpha}\cos z\,\mathrm{d}z$ exists and is finite, and

$$2\int_0^\infty z^{\gamma-\alpha}\cos z\,\mathrm{d}z = 2\Gamma(\gamma-\alpha+1)\cos\frac{(\gamma-\alpha+1)\pi}{2} =: c_1(\alpha,\gamma). \tag{3.35}$$

As a consequence,

$$\lim_{M\to\infty}\int_0^M z^{\gamma-\alpha}\cos z\,\mathrm{d}z = \int_0^\infty z^{\gamma-\alpha}\cos z\,\mathrm{d}z = \frac{1}{2}\,c_1(\alpha,\gamma),$$

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from which it easily follows that

$$\left|\int_0^{N|\xi|} z^{\gamma-\alpha}\cos z\,\mathrm{d}z\right| \leqslant c,$$

where $c$ is a constant independent of $N$ and $\xi$. Using (3.32) and (3.34), this implies

$$|c(N,\xi,\gamma,\alpha)\,F\psi_n(\xi)| \leqslant 2c|\xi|^{\alpha-(1+\gamma)}|F\psi_n(\xi)|,$$

where the right-hand side is an integrable function in view of $\alpha-(1+\gamma)>-1$ and $F\psi_n \in S(\mathbf{R})$. An application of Lebesgue's dominated convergence theorem now shows that the right-hand side of (3.33) converges to

$$F^{-1}[c_1(\alpha,\gamma)|\xi|^{\alpha-(1+\gamma)}F\psi_n](x) = c_1(\alpha,\gamma)F^{-1}[|\xi|^{\alpha-(1+\gamma)}F\psi_n](x), \qquad x\in\mathbf{R},$$

as $N\to\infty$. By monotone convergence, the left-hand side of (3.33) converges to $|x|^{\gamma-\alpha} * \psi_n(x)$ as $N\to\infty$. This proves Lemma 3.4.

Now the proof of Proposition 3.2 can be easily completed. Using assertions (a) and (b) of Lemma 3.3, identities (3.28) and (3.29), and Lebesgue's dominated convergence, this gives

$$\begin{aligned}\mathscr{G}V_n &= \lim_{\varepsilon\downarrow 0} -\frac{1}{2}\gamma\Gamma(\gamma)F^{-1}\left[|\xi|^{\alpha-(1+\gamma)}\left(\frac{|\xi|^{\gamma+1}}{(\varepsilon+i\xi)^{\gamma+1}}+\frac{|\xi|^{\gamma+1}}{(\varepsilon-i\xi)^{\gamma+1}}\right)F\psi_n\right]\\ &= -\Gamma(\gamma+1)\cos\left(\frac{\gamma+1}{2}\pi\right)F^{-1}[|\xi|^{\alpha-(1+\gamma)}\,F\psi_n].\end{aligned}$$

Finally, taking into account Lemma 3.4, we obtain (3.26) with

$$c(\alpha,\gamma) = -\frac{\Gamma(\gamma+1)\cos((\gamma+1)\pi/2)}{c_1(\alpha,\gamma)}.$$

This proves (3.26). The proof of assertion (b) of Proposition 3.2 follows along the same lines as the proof in assertion (a), so that we omit the details. Proposition 3.2 is proved.

**IV.** In this final part of the proof of Theorem 2.1, we consider the limit behavior of the second part on the right-hand side of relation (3.4), which is denoted by $I_t^n$ (see (3.6)).

PROPOSITION 3.3. (a) *For all* $0<\alpha<2$ *and* $\alpha-1<\gamma<\alpha$,

$$\lim_{n\to\infty} I_t^n = c(\alpha,\gamma)\int_0^t |Z_s-a|^{\gamma-\alpha}\,\mathrm{d}s, \qquad t\geqslant 0, \tag{3.36}$$

*in* $L^1(\mathbf{P})$ *and in probability.*

(b) *In the boundary case, when* $\gamma=\alpha-1$ (*thus* $1<\alpha<2$), *it holds for all* $t>0$ *that*

$$\lim_{n\to\infty} I_t^n = c(\alpha)\,L^Z(t,a), \qquad t\geqslant 0, \tag{3.37}$$

*in* $L^1(\mathbf{P})$ *and in probability, where* $L^Z(t,a)$ *is the local time of* $Z$ *up to time* $t$ *in* $a$.

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*Proof.* First, we note that the probability measures $\rho_n(dx) := \psi_n(x)\,\mathrm{d}x$ are supported by $[-1,1]$ and converge weakly to the Dirac measure $\delta_0$ at 0. Therefore,

$$\lim_{n\to\infty} c(\alpha,\gamma)|x|^{\gamma-\alpha} * \psi_n(x) = c(\alpha,\gamma)|x|^{\gamma-\alpha}, \qquad x \in \mathbf{R}.$$

Using Proposition 3.2(a), this yields

$$\begin{aligned}\lim_{n\to\infty} \mathscr{G}V_n(x) &= \lim_{n\to\infty}\lim_{\varepsilon\downarrow 0} \mathscr{A}V_{\varepsilon,n}(x)\\ &= \lim_{n\to\infty} c(\alpha,\gamma)|x|^{\gamma-\alpha} * \psi_n(x) = c(\alpha,\gamma)|x|^{\gamma-\alpha}.\end{aligned}$$

Next, we show that the sequence $(\mathscr{G}V_n(Z_s-a))_{n\geqslant 1}$ is uniformly integrable with respect to the product measure $\lambda_+ \otimes \mathbf{P}$ on $[0,t]\times\Omega$. In view of the theorem of de la Vallée Poussin (cf. [9, Theorem II.22]), for this it suffices to verify that for some $p>1$ we have

$$\mathbf{E}\bigg[\int_0^t \big(|Z_s-a|^{\gamma-\alpha} * \psi_n(x)\big)^p\,\mathrm{d}s\bigg] \leqslant c < \infty.$$

Note that $0<\alpha-\gamma<\alpha\wedge 1$. Fixing $p>1$ such that $0<p(\alpha-\gamma)<\alpha\wedge 1$, and taking into account Jensen's inequality and Fubini's theorem, we can estimate

$$\begin{aligned}\mathbf{E}\bigg[\int_0^t \big(|Z_s-a|^{\gamma-\alpha} * \psi_n(x)\big)^p\,\mathrm{d}s\bigg] &= \mathbf{E}\bigg[\int_0^t \bigg(\int_{\mathbf{R}} |Z_s-a-y|^{\gamma-\alpha}\psi_n(y)\,\mathrm{d}y\bigg)^p \mathrm{d}s\bigg]\\ &\leqslant \mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}} |Z_s-a-y|^{p(\gamma-\alpha)}\psi_n(y)\,\mathrm{d}y\,\mathrm{d}s\bigg]\\ &= \int_{\mathbf{R}} \mathbf{E}\bigg[\int_0^t |Z_s-a-y|^{p(\gamma-\alpha)}\,\mathrm{d}s\bigg]\psi_n(y)\,\mathrm{d}y \leqslant c\int_{\mathbf{R}}\psi_n(y)\,\mathrm{d}y = c,\end{aligned}$$

where we have used Proposition 1.4 justified by the relation $0<p(\alpha-\gamma)<\alpha\wedge 1$.

By definition, we have

$$I_t^n = \int_0^t \mathscr{G}V_n(Z_s-a)\,\mathrm{d}s$$

(see (3.6) and (1.8)). We have shown above that the integrand $\mathscr{G}V_n(Z_s-a)$ converges to $c(\alpha,\gamma)|Z_s-a|^{\gamma-\alpha}$ as $n\to\infty$ pointwise and is uniformly integrable with respect to the product measure $\lambda_+\otimes\mathbf{P}$ on $[0,t]\times\Omega$. According to [2, Proposition 21.4] it follows that $\mathscr{G}V_n(Z_s-a)$ converges to $c(\alpha,\gamma)|Z_s-a|^{\gamma-\alpha}$ in $L^1([0,t]\times\Omega,\lambda_+\otimes\mathbf{P})$, and, in particular, that

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}\bigg[\bigg|I_t^n - \int_0^t c(\alpha,\gamma)|Z_s-a|^{\gamma-\alpha}\,\mathrm{d}s\bigg|\bigg]&\\ \leqslant \lim_{n\to\infty}\mathbf{E}\bigg[\int_0^t \big|\mathscr{G}V_n(Z_s-a) - c(\alpha,\gamma)|Z_s-a|^{\gamma-\alpha}\big|\,\mathrm{d}s\bigg] &= 0.\end{aligned}$$

This implies (3.36), and hence the proof of Proposition 3.3(a) is complete.

The proof of part (b) of Proposition 3.3 is much simpler. Using Proposition 3.2(b), and the occupation times formula for the local time $L^Z(t,a)$, we obtain

$$\begin{aligned}I_t^n &= \int_0^t \mathscr{G}V_n(Z_s-a)\,\mathrm{d}s = \int_0^t c(\alpha)\psi_n(Z_s-a)\,\mathrm{d}s\\ &= c(\alpha)\int_{\mathbf{R}}\psi_n(z-a)L^Z(t,z)\,\mathrm{d}z = c(\alpha)\int_{\mathbf{R}} L^Z(t,z+a)\psi_n(z)\,\mathrm{d}z.\end{aligned}$$

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Since the probability measures $\rho_n(\mathrm{d}z) = \psi_n(z)\,\mathrm{d}z$ weakly converge to the Dirac measure $\delta_0$ at point 0, and the local time $L^Z(t,z)$ is continuous and bounded in the state variable $z$ **P**-a.s., we see that the right-hand side converges to $c(\alpha)L^Z(t,a)$ **P**-a.s. This completes the proof of Proposition 3.3.

To finish the proof of Theorem 2.1, it remains to make $n \to \infty$ in (3.4) and to use Propositions 3.1, 3.2, and 3.3.

**Appendix A. Itô's formula.** For the convenience of the reader, the Itô formula (1.2) for symmetric stable processes is provided. To this end, the Itô formula for semimartingales (cf., e.g., [10, Theorem II.33]) is taken for granted.

Let $Z$ be a Lévy process with characteristic triplet $(\beta, \sigma, \nu)$ (with respect to the standard truncation function). For the sake of simplicity, it is assumed that $\beta = 0$ and $\sigma = 0$. The Itô–Lévy decomposition yields

$$Z_t = \int_0^t \int_{\{|y|\leqslant 1\}} y\, q(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{\{|y|>1\}} y\, \mu(\mathrm{d}s, \mathrm{d}y), \tag{A.1}$$

where $\mu$ is the Poisson random measure associated with $Z$, $m = \mathbf{E}[\mu]$ is its intensity measure, and $q = \mu - m$ is the compensated Poisson random measure. Obviously, the process $Z$ is a semimartingale. Let $g \in C^2(\mathbf{R})$. Using the Itô formula for semimartingales (cf., e.g., [10, Theorem II.33]) and noting that the continuous martingale part $Z^c$ of the semimartingale $Z$ vanishes, yields

$$\begin{aligned} g(Z_t) = g(Z_0) &+ \int_0^t g'(Z_{s-})\,\mathrm{d}Z_s + \frac{1}{2}\int_0^t g''(Z_{s-})\,\mathrm{d}[Z^c, Z^c]_s \\ &+ \sum_{0<s<t} \big(g(Z_s) - g(Z_{s-}) - g'(Z_{s-})\,\Delta Z_s\big) \end{aligned} \tag{A.2}$$

$$= g(Z_0) + \int_0^t g'(Z_{s-})\,\mathrm{d}Z_s + \sum_{0<s\leqslant t} \big(g(Z_s) - g(Z_{s-}) - g'(Z_{s-})\,\Delta Z_s\big). \tag{A.3}$$

Calculating

$$\begin{aligned} &\int_0^t g'(Z_{s-})\,\mathrm{d}Z_s \\ &\quad= \int_0^t g'(Z_{s-}) \int_{\{|y|\leqslant 1\}} y\, q(\mathrm{d}s, \mathrm{d}y) + \int_0^t g'(Z_{s-}) \int_{\{|y|>1\}} y\, \mu(\mathrm{d}s, \mathrm{d}y) \\ &\quad= \int_0^t \int_{\{|y|\leqslant 1\}} g'(Z_{s-}) y\, q(\mathrm{d}s, \mathrm{d}y) + \int_0^t \int_{\{|y|>1\}} g'(Z_{s-}) y\, \mu(\mathrm{d}s, \mathrm{d}y) \end{aligned}$$

and inserting this expression in (A.3) implies

$$\begin{aligned} g(Z_t) &= g(Z_0) + \int_0^t \int_{\{|y|\leqslant 1\}} g'(Z_{s-}) y\, q(\mathrm{d}s, \mathrm{d}y) \\ &\qquad + \sum_{0<s\leqslant t} g'(Z_{s-})\Delta Z_s \mathbf{1}_{\{|\Delta Z_s|>1\}} \\ &\qquad + \sum_{0<s\leqslant t} \big(g(Z_s) - g(Z_{s-}) - g'(Z_{s-})\Delta Z_s\big) \\ &= g(Z_0) + \int_0^t \int_{\{|y|\leqslant 1\}} g'(Z_{s-}) y\, q(\mathrm{d}s, \mathrm{d}y) \end{aligned}$$

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$$
\begin{aligned}
&+\int_0^t\int_{\mathbf{R}}\big(g(Z_{s-}+y)-g(Z_{s-})-\mathbf{1}_{\{|y|\leqslant 1\}}g'(Z_{s-})y\big)\,\mu(\mathrm{d}s,\mathrm{d}y)\\
&=g(Z_0)+\int_0^t\int_{\{|y|\leqslant 1\}}g'(Z_{s-})y\,q(\mathrm{d}s,\mathrm{d}y)\\
&\quad+\int_0^t\int_{\mathbf{R}}\big(g(Z_{s-}+y)-g(Z_{s-})-\mathbf{1}_{\{|y|\leqslant 1\}}g'(Z_{s-})\,y\big)\,q(\mathrm{d}s,\mathrm{d}y)\\
&\quad+\int_0^t\int_{\mathbf{R}}\big(g(Z_{s-}+y)-g(Z_{s-})-\mathbf{1}_{\{|y|\leqslant 1\}}g'(Z_{s-})y\big)\,m(\mathrm{d}s,\mathrm{d}y)\\
&=g(Z_0)+\int_0^t\int_{\mathbf{R}}\big(g(Z_{s-}+y)-g(Z_{s-})\big)\,q(\mathrm{d}s,\mathrm{d}y)\\
&\quad+\int_0^t\int_{\mathbf{R}}\big(g(Z_{s-}+y)-g(Z_{s-})-\mathbf{1}_{\{|y|\leqslant 1\}}g'(Z_{s-})\,y\big)\,m(\mathrm{d}s,\mathrm{d}y).
\end{aligned}\tag{A.4}
$$

Hence the Itô formula (1.9) is shown.

*Remark* A.1. The integrals in the equality (A.4) indeed exist.

Let us first consider the case when $g$, $g'$, and $g''$ are bounded. Then we get

$$
\begin{aligned}
&\mathbf{E}\bigg[\int_0^t\int_{\mathbf{R}}\big(g(Z_{s-}+y)-g(Z_{s-})\big)^2\mathbf{1}_{\{|y|\leqslant 1\}}\,m(\mathrm{d}s,\mathrm{d}y)\bigg]\\
&\qquad\leqslant c^2\int_0^t\int_{\{|y|\leqslant 1\}}y^2\,m(\mathrm{d}s,\mathrm{d}y)<+\infty,
\end{aligned}
$$

where $c$ is an upper bound of $|g'|$, and

$$
\mathbf{E}\bigg[\int_0^t\int_{\{|y|>1\}}|g(Z_{s-}+y)-g(Z_{s-})|\,m(\mathrm{d}s,\mathrm{d}y)\bigg]\leqslant ct\nu([-1,1]^c)<+\infty,
$$

where $c$ is an upper bound of $2|g|$, meaning that the first integral exists and is a martingale. Similarly, the second integral exists and is finite $\mathbf{P}$-a.s. For this, we note that on $\{|y|\leqslant 1\}$

$$
|g(Z_{s-}+y)-g(Z_{s-})-g'(Z_{s-})y|\leqslant cy^2,
$$

where $c$ is an upper bound of $|g''|$.

In the general case, we have to introduce the stopping times $T_N$ by

$$
T_N:=\inf\{t\geqslant 0\colon |Z_{t-}|\geqslant N\},\qquad N\geqslant 1.
$$

Then it follows that $T_N\uparrow\infty$ as $N\to\infty$ and $|Z_{(T_N\wedge t)-}|\leqslant N$, and the functions $g$, $g'$, and $g''$ are bounded on $[-N,N]$. As above, it can be shown that the first process in equality (A.4) stopped at $T_N$ is well defined and is a martingale, and the second process in equality (A.4) stopped at $T_N$ exists and is finite $\mathbf{P}$-a.s.

Hence the first integral is a local martingale, and the second is a process of locally bounded variation. Formula (A.4) gives the semimartingale decomposition of $g(Z)$.

## Appendix B. Some inequalities.

PROPOSITION B.1. (a) *For all* $0<\gamma\leqslant 1$ *and* $x,y\in\mathbf{R}$,

$$
|x+y|^\gamma\leqslant|x|^\gamma+|y|^\gamma.\tag{B.1}
$$

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(b) *For all* $0 < \gamma \leqslant 2$ *and* $x, y \in \mathbf{R}$,

$$|x+y|^\gamma \leqslant 2(|x|^\gamma + |y|^\gamma). \tag{B.2}$$

*Proof.* First we verify (a). Let $0 < \gamma \leqslant 1$. The proof is divided into six cases.

*Case* 1: $x, y \geqslant 0$. Let $x \geqslant 0$ be fixed. We define the functions $f$ and $g$ by

$$f(y) = (x+y)^\gamma \quad \text{and} \quad g(y) = x^\gamma + y^\gamma$$

for $y \geqslant 0$. Obviously, $f(0) = g(0)$. Furthermore, for $y > 0$,

$$f'(y) = \gamma\,(x+y)^{\gamma-1} \quad \text{and} \quad g'(y) = \gamma\, y^{\gamma-1},$$

and in view of $\gamma - 1 \leqslant 0$ we see that $f'(y) \leqslant g'(y)$ for all $y > 0$. Hence $f(y) \leqslant g(y)$ for all $y \geqslant 0$, and the claim is proved.

*Case* 2: $x \leqslant 0$ and $x + y \geqslant 0$. When this case holds, then $0 \leqslant x + y \leqslant y$, and hence $|x+y|^\gamma \leqslant |y|^\gamma$ so that the asserted inequality holds.

*Case* 3: $y \geqslant 0$ and $x + y \leqslant 0$ (hence $x \leqslant 0$). This case can be treated by introducing the new variables $x' = -x$ and $y' = -y$ and therefore reduces to the case that $x + y \geqslant 0$ and $y \leqslant 0$ (hence $x \geqslant 0$). After changing the roles of $x$ and $y$, this is just Case 2.

*Case* 4: $x, y \leqslant 0$. This can be reduced to $x, y \geqslant 0$ (Case 1) by setting $x' = -x$ and $y' = -y$.

*Case* 5: $y \leqslant 0$ and $x + y \geqslant 0$ (hence $x \geqslant 0$). This is Case 2 after changing the roles of $x$ and $y$.

*Case* 6: $y \leqslant 0$ and $x + y \leqslant 0$. This is equivalent to $y \geqslant 0$ and $x + y \geqslant 0$ which is Case 1 if $x \geqslant 0$ and Case 2 if $x \leqslant 0$.

This proves (a).

Part (b) is a simple application of (a) for the exponent $\gamma/2$ which satisfies $0 < \gamma/2 \leqslant 1$:

$$|x+y|^\gamma = (|x+y|^2)^{\gamma/2} \leqslant (2x^2 + 2y^2)^{\gamma/2} \leqslant 2^{\gamma/2}(x^{2\,\gamma/2} + y^{2\,\gamma/2}) \leqslant 2(|x|^\gamma + |y|^\gamma).$$

The proof of Proposition B.1 is complete.

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