# The Effect of Asynchronous Execution and Message Latency on Max-sum

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## Abstract

Max-sum is a version of belief propagation that was adapted for solving distributed constraint optimization problems (DCOPs). It has been studied theoretically and empirically, extended to versions that improve solution quality and converge rapidly, and is applicable to multiple distributed applications. The algorithm was presented both as a synchronous and an asynchronous algorithm, however, neither the differences in the performance of these two execution versions nor the implications of message latency on the two versions have been investigated to the best of our knowledge.

We contribute to the body of knowledge on Max-sum by: (1) Establishing the theoretical differences between the two execution versions of the algorithm, focusing on the construction of beliefs; (2) Empirically evaluating the differences between the solutions generated by the two versions of the algorithm, with and without message latency; and (3) Establishing both theoretically and empirically the positive effect of damping on reducing the differences between the two versions. Our results indicate that in contrast to recent published results indicating the drastic effect that message latency has on distributed local search, damped Max-sum is robust to message latency.

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## 1 Introduction

Recent advances in computation and communication have resulted in realistic distributed applications, in which humans and technology interact and aim to optimize mutual goals (e.g., IoT applications). A promising multi-agent approach to solve these types of problems is to model them as *distributed constraint optimization problems* (DCOPs), where decision makers are modeled as *agents* that assign *values* to their *variables*. The goal in a DCOP is to optimize a global objective in a decentralized manner. Unfortunately, the communication assumptions of the DCOP model are overly simplistic and often unrealistic: (1) All messages arrive *instantaneously* or have *very small and bounded delays*; and (2) Messages sent *arrive in the order that they were sent*. These assumptions do not reflect real-world characteristics, where messages may be disproportionally delayed due to different bandwidths in different communication channels.

Recently, a study that investigated the effect of message latency on standard distributed local search algorithms, e.g., MGM and DSA, has shown that message delays have a *a dramatic positive*


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*effect* on the performance of the asynchronous versions of these algorithms [18]. Apparently, message latency generates an exploration effect, which improves significantly the quality of the solutions they produce. Nevertheless, this study did not investigate the effect on distributed incomplete inference algorithms, e.g., the Max-sum algorithm, although, these algorithms have been shown recently to be most successful [3, 4]. Thus, we focus our attention to the effect of message latency on Max-sum and its variants in this paper.

Max-sum is a version of the belief propagation algorithm [16, 25], which is used for solving DCOPs. It has been recently proposed for solving multi-agent optimization problems in applications, such as sensor systems [23, 22], task allocation for rescue teams in disaster areas [19], and smart homes [21]. As with most belief propagation algorithms, Max-sum is known to converge to an optimal solution when solving problems represented by acyclic graphs. On problems represented by cyclic graphs, the beliefs may fail to converge, and the resulting assignments that are considered optimal under those beliefs may be of low quality [6, 30]. This occurs because cyclic information propagation leads to computation of inaccurate and inconsistent information [16].

To decrease the effect of cyclic information propagation in belief propagation, the *damping* method has been suggested. It balances the weight of the new calculation performed in each iteration and the weight of calculations performed in previous iterations, resulting in an increased probability for convergence [4]. Recently, splitting nodes in the factor graph on which belief propagation operates has been shown to be an effective method for accelerating the convergence of the algorithm when combined with damping [20, 4].

Max-sum has been presented both as an asynchronous and as a synchronous algorithm (e.g., [6, 30, 5]). In the synchronous version, agents perform in iterations. In each iteration, an agent sends messages to all its neighbors and waits for the messages sent to it from all its neighbors to arrive, before moving to the next iteration. In the asynchronous version, agents react to messages when they arrive. To best of our knowledge, the implications of this difference in the execution of the algorithm on its performance have not been studied to date. Moreover, while message latency does not affect the actions that agents perform (only delays them) in the synchronous version, intuitively, it is expected to have a major effect on the performance of the asynchronous version. The reason is that the beliefs included in messages are used by agents in the construction of beliefs that they propagate to others and in their assignment selection. In asynchronous execution, belief construction and assignment selection might be performed while considering imbalanced and inconsistent information.

In this paper, we make the following contributions:

1. We analyze the properties of the two execution versions of Max-sum, synchronous and asynchronous. More specifically, using backtrack cost trees [28], we investigate the possible differences between the propagated beliefs in synchronous and asynchronous executions of Max-sum.
2. We investigate the effect of damping on asynchronous Max-sum. While there are clear indications (both empirical and theoretical) that damping improves the performance of the synchronous version of Max-sum [4, 28], to best of our knowledge, the effect of damping on the asynchronous version of Max-sum has not been studied. We analyze this effect both theoretically and empirically. Both indicate that damping reduces the differences between synchronous and asynchronous execution.
3. We investigate the performance of the different versions of the algorithm in the presence of message latency. While the beliefs propagated and the computation that agents perform are not affected by message latency in the synchronous version (only delayed), this is not true for the asynchronous version. Once again, our empirical results reveal that damping reduces the differences. Moreover, the version of Max-sum proposed by [4] that includes both damping and splitting maintains its fast convergence properties and the quality of solutions, even in asynchronous execution with message delays.

# 2 Background

In this section we provide background on *graphical models*, *distributed constraint optimization problems* (DCOPs), the DCOP versions of belief propagation – *Max-sum* and its variants – and *backtrack cost tree* (BCT) – the tool we use to analyze the algorithms' behavior. While the Max-sum variants that we discuss are actually solving a min-sum problem [20], we will still refer to them as "Max-sum" since this name is commonly used [6, 7, 30].

## 2.1 Graphical Models

Graphical models such as Bayesian networks or constraint networks are a widely used representation framework for reasoning and solving optimization problems. The graph structure is used to capture dependencies between variables [11]. Our work extends the theory established in [24], which considered the most a priori Maximum a posteriori (MAP) assignment, which is solved using the Max-product version of belief propagation. The relation between MAP and constraint optimization is well established [11, 6, 15], and thus, results that consider Max-product for MAP apply to Max/Min-sum for solving constraint optimization problems, as well as the other way round [20]. Without loss of generality, we will focus on constraint optimization, since it is more common in AI literature. Moreover, we will consider the distributed version of the problem, since it is a natural representation for message passing algorithms. Nevertheless, our results apply to any version of problem represented by a graphical model and solved by belief propagation, as do the results of [24].

## 2.2 Distributed Constraint Optimization Problems

Without loss of generality, in the rest of this paper, we will assume that all problems are minimization problems, as it is common in the DCOP literature (e.g., [13]). Thus, we assume that all constraints define costs and not utilities.

A DCOP is defined by a tuple $\langle \mathcal{A}, \mathcal{X}, \mathcal{D}, \mathcal{R} \rangle$. $\mathcal{A}$ is a finite set of agents $\{A_1, A_2, \dots, A_n\}$. $\mathcal{X}$ is a finite set of variables $\{X_1, X_2, \dots, X_m\}$. Each variable is held by a single agent, and an agent may hold more than one variable. $\mathcal{D}$ is a set of domains $\{D_1, D_2, \dots, D_m\}$. Each domain $D_i$ contains the finite set of values that can be assigned to variable $X_i$. We denote an assignment of value $x \in D_i$ to $X_i$ by an ordered pair $\langle X_i, x \rangle$. $\mathcal{R}$ is a set of relations (constraints). Each constraint $R_j \in \mathcal{R}$ defines a non-negative *cost* for every possible value combination of a set of variables, and is of the form $R_j : D_{j_1} \times D_{j_2} \times \dots \times D_{j_k} \rightarrow \mathbb{R}^+ \cup \{0\}$. A *binary constraint* refers to exactly two variables and is of the form $R_{ij} : D_i \times D_j \rightarrow \mathbb{R}^+ \cup \{0\}$.[1] For each binary constraint $R_{ij}$, there is a corresponding cost table $T_{ij}$ with dimensions $|D_i| \times |D_j|$ in which the cost in every entry $e_{xy}$ is the cost incurred when $X_i$ is assigned to $x$ and $X_j$ is assigned to $y$. A *binary DCOP* is a DCOP in which all constraints are binary. A *partial assignment* is a set of value assignments to variables, in which each variable appears at most once. $vars(PA)$ is the set of all variables that appear in partial assignment $PA$, i.e., $vars(PA) = \{X_i \mid \exists x \in D_i \wedge \langle X_i, x \rangle \in PA\}$. A constraint $R_j \in \mathcal{R}$ of the form $R_j : D_{j_1} \times D_{j_2} \times \dots \times D_{j_k} \rightarrow \mathbb{R}^+ \cup \{0\}$ is *applicable* to $PA$ if each of the variables $X_{j_1}, X_{j_2}, \dots, X_{j_k}$ is included in $vars(PA)$. The *cost of a partial assignment* $PA$ is the sum of all applicable constraints to $PA$ over the value assignments in $PA$. A *complete assignment* (or a *solution*) is a partial assignment that includes all the DCOP's variables (i.e., $vars(PA) = \mathcal{X}$). An *optimal solution* is a complete assignment with minimal cost.

[1] We say that a variable is *involved* in a constraint if it is one of the variables the constraint refers to.

For simplicity, we make the common assumption that each agent holds exactly one variable (i.e., $n = m$) and we concentrate on binary DCOPs. These assumptions are common in the DCOP literature (e.g., [17, 26]). In addition to the standard motivation for focusing on binary DCOPs, in the case of Max-sum it is essential, since the runtime complexity of each iteration of Max-sum is exponential in the arity of the constraints.

## 2.3 The Max-Sum Algorithm

Max-sum operates on a *factor graph*, which is a bipartite graph in which the nodes represent variables and constraints [10]. Each variable-node representing a variable of the original DCOP is connected to all function-nodes representing constraints that it is involved in. Similarly, a function-node is connected to all variable-nodes representing variables in the original DCOP that are involved in it. Variable-nodes and function-nodes are considered "agents" in Max-sum (i.e., they can send and receive messages, and can perform computation).

A message sent to or from variable-node $X$ (for simplicity, we use the same notation for a variable and the variable-node representing it) is a vector of size $|D_X|$ including a cost for each value in $D_X$. These costs are also called *beliefs*. Before the first iteration, all nodes assume that all messages they previously received (in iteration 0) include vectors of zeros. A message sent from a variable-node $X$ to a function-node $F$ in iteration $i \geq 1$ is formalized as follows:

$$Q^i_{X\to F} = \sum_{F' \in F_X, F' \neq F} R^{i-1}_{F'\to X} - \alpha \tag{1}$$

where $Q^i_{X\to F}$ is the message variable-node $X$ intends to send to function-node $F$ in iteration $i$, $F_X$ is the set of function-node neighbors of variable-node $X$, and $R^{i-1}_{F'\to X}$ is the message sent to variable-node $X$ by function-node $F'$ in iteration $i-1$. $\alpha$ is a constant that is reduced from all beliefs included in the message (i.e., for each $x \in D_X$) in order to prevent the costs carried by messages throughout the run of the algorithm from growing arbitrarily large.

A message $R^i_{F\to X}$ sent from a function-node $F$ to a variable-node $X$ in iteration $i$ includes for each value $x \in D_X$:

$$min_{PA_{-X}} cost(\langle X, x\rangle, PA_{-X}) \tag{2}$$

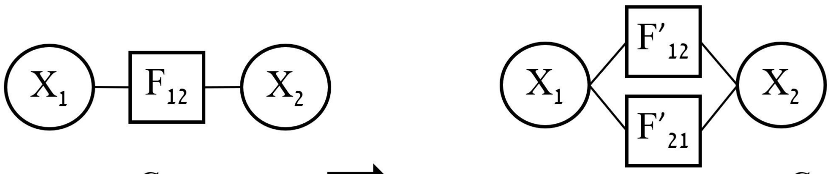
where $PA_{-X}$ is a possible combination of value assignments to variables involved in $F$ not including $X$. The term $cost(\langle X, x\rangle, PA_{-X})$ represents the cost of a partial assignment $a = \{\langle X, x\rangle, PA_{-X}\}$, which is:

$$f(a) + \sum_{X' \in X_F, X' \neq X, \langle X', x'\rangle \in a} (Q^{i-1}_{X'\to F})_{x'} \tag{3}$$

where $f(a)$ is the original cost in the constraint represented by $F$ for the partial assignment $a$, $X_F$ is the set of variable-node neighbors of $F$, and $(Q^{i-1}_{X'\to F})_{x'}$ is the cost that was received in the message sent from variable-node $X'$ in iteration $i-1$, for the value $x'$ that is assigned to $X'$ in $a$. $X$ selects its value assignment $\hat{x} \in D_X$ following iteration $k$ as follows:

$$\hat{x} = \arg\min_{x \in D_X} \sum_{F \in F_X} (R^k_{F\to X})_x \tag{4}$$

In the synchronous version (*Syn_Max-sum*), at each iteration $t$, an agent waits to receive all messages sent to it in iteration $t-1$ before performing computation and generating the messages to be sent in that iteration [30]. In the asynchronous version (*Asy_Max-sum*), agents react to messages they receive. Whenever a node receives a message, it performs computation and sends out messages




$C_{F_{12}}$

| $X_2$ / $X_1$ | a | b |
|---|---|---|
| a | 120 | 190 |
| b | 360 | 390 |

$C_{F'_{12}}$

| $X_2$ / $X_1$ | a | b |
|---|---|---|
| a | 11 | 127 |
| b | 352 | 14 |

$C_{F'_{21}}$

| $X_2$ / $X_1$ | a | b |
|---|---|---|
| a | 109 | 63 |
| b | 8 | 376 |

**Figure 1** An acyclic DCOP factor graph (on the left) and its equivalent SCFG (on the right).

to its neighbors, taking into consideration the last message received from each of its neighbors [6]. In both versions, the logic for the actions of the agents are identical, only the trigger for performing those actions is different.

### 2.3.1 Damped Max-sum (DMS)

DMS has an additional feature, which is the damping of the propagated beliefs. In order to add damping to Max-sum, a parameter $\lambda \in [0, 1)$ is used. Before sending a message in iteration $k$, an agent performs calculations as in standard Max-sum. We use $\widehat{m^k_{i \to j}}$ to denote the result of the calculation made by agent $A_i$ for the content of a message intended to be sent from $A_i$ to agent $A_j$ in iteration $k$ and $m^{k-1}_{i \to j}$ to denote the message sent by $A_i$ to $A_j$ at iteration $k - 1$. The message sent by $A_i$ to $A_j$ at iteration $k$ is calculated as follows:

$$m^k_{i \to j} = \lambda m^{k-1}_{i \to j} + (1 - \lambda)\widehat{m^k_{i \to j}} \tag{5}$$

Thus, $\lambda$ expresses the weight given to previously performed calculations with respect to the most recent calculation performed. Moreover, when $\lambda = 0$ the resulting algorithm is standard Max-sum.

We use Syn_DMS and Asy_DMS to denote the synchronous and asynchronous versions of DMS, respectively, in this paper.

### 2.3.2 Asynchronous Execution

All the definitions used for describing Max-sum (and DMS) above use the iteration number $k$. It was used to describe how a message is generated, using the information received by the factor graph node in the previous iteration ($k - 1$). In asynchronous execution, their are no iterations, and agents perform computation steps whenever they receive messages. Thus, in asynchronous execution, the information that a node $N_i$ uses, when it generates a message in some time $t$, is, for each neighbor $N_j$, the information included in the last message received from $N_j$ (prior to $t$), regardless of when it was sent by $N_j$. If no message has been received from $N_j$ yet, $N_i$ uses a vector of zeros in its computation. Notice, that in the presence of message delays, a node $N_i$ may receive messages from its neighbor $N_j$, not in the order they were sent. This is true for both the synchronous and the asynchronous versions of the algorithm. Nevertheless, the agents use the messages in the order in which they were received.

In order to avoid this phenomenon, we implemented a time-stamp method that allowed the agents receiving messages to consider the information they include in the order that they were sent. However, the results were not significantly different from the results obtained when we did not use this method, thus, we do not report these results in our empirical study.

### 2.3.3 Max-sum with Split Constraint Factor Graphs

When Max-sum is applied to an asymmetric problem, the representing factor graph has each (binary) constraint represented by two function-nodes, one for each part of the constraint held by one of the involved agents. Each function-node is connected to both variable-nodes representing the variables involved in the constraint [31]. Figure 1 presents two equivalent factor graphs that include two variable-nodes, each with two values in its domain, and a single binary constraint. On the left, the factor graph represents a (symmetric) DCOP including a single constraint between variables $X_1$ and $X_2$, hence, it includes a single function node representing this constraint. On the right, the equivalent factor graph representing the equivalent asymmetric DCOP is depicted. It includes two function-nodes, representing the parts of the constraint held by the two agents involved in the asymmetric constraint. Thus, the cost table in each function-node includes the asymmetric costs that the agent holding this function-node incurs. In this example function-node $F'_{12}$ is held by agent $A_1$, while $F'_{21}$ is held by $A_2$. The factor graphs are equivalent since the sum of the two cost tables held by the function-nodes representing the constraints in the factor graph on the right, is equal to the cost table of the single function-node representing this constraint in the factor graph on the left (see [32] for details). Researchers have used such *Split Constraint Factor Graphs* (SCFGs) as an enhancement method for Max-sum [20, 4]. This is achieved by splitting each constraint that was represented by a single function-node in the original factor graph into two function-nodes. The SCFG is equivalent to the original factor graph if the sum of the cost tables of the two function-nodes representing each constraint in the SCFG is equal to the cost table of the single function-node representing the same constraint in the original factor graph. By tuning the similarity between the two function-nodes representing the same constraint one can determine the level of asymmetry in the SCFG. The use of symmetric SCFGs was shown to trigger very fast convergence to high quality solutions. However, generating mild asymmetry, postpones convergence and generates some exploration, which results in improved solution quality [4].

### 2.3.4 Non-Concurrent Logic Operations

In order to evaluate the performance of distributed algorithms performing in a distributed environment, there is a need to establish which of the operations performed by agents could not have been performed concurrently and, thus, the run-time performance of the algorithm is the longest non-concurrent sequence of operations that the algorithm performed. In [29], DisCSP algorithms were evaluated, which their basic logic operations were constraint checks (CCs), thus, the performance was measured in terms of non-concurrent constraint checks (NCCCs). In [14], search based complete algorithms were compared with inference algorithms, thus, algorithms that perform different atomic logic operations (i.e., constraint checks and compatibility checks) were compared, and the results were reported in terms of non-concurrent logic operations (NCLOs). This approach is the one we adopt in this study, since we evaluate the quality of the solutions of the algorithms, as a function of the asynchronous advancement of the algorithm, when agents perform computation concurrently.

Recently, these insights were generalized such that similar statements can be made when the algorithm is solving finite factor-graphs with multiple cycles [28]. Zivan *et al.* have proved that, as in the single cycle case, on every finite factor-graph, Max-sum at some point in time starts to repeatedly follow a path that minimizes its beliefs. When a large enough damping factor is used, this minimal path is indeed the minimal path in the factor-graph, and thus, if it is consistent, the algorithm converges to the optimal solution.

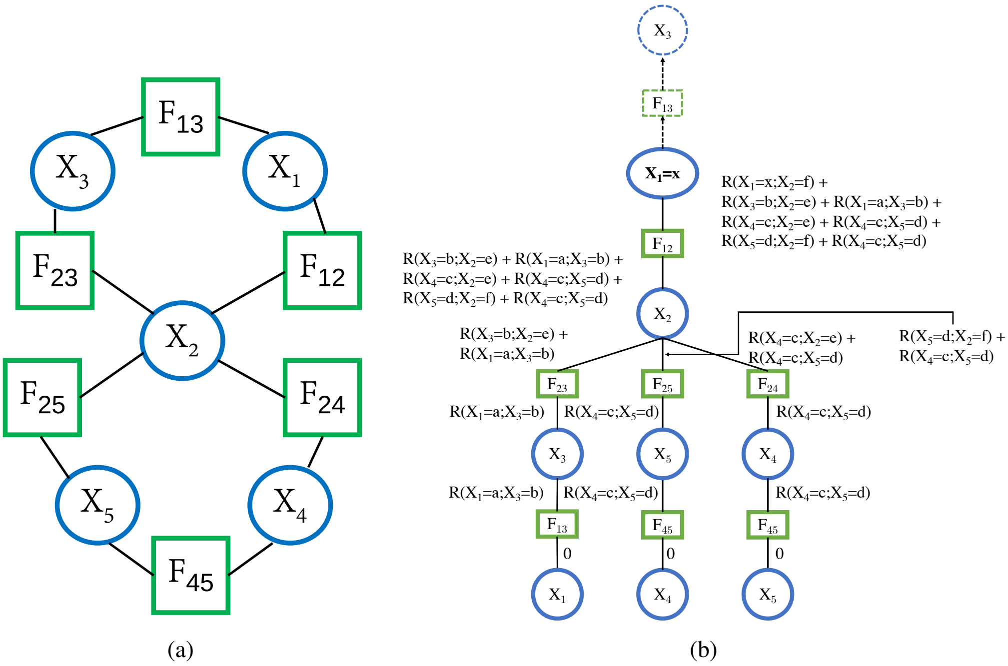



**Figure 2** (a) A lemniscate factor-graph. (b) An example of a BCT for a belief in the message sent from $X_1$ to the function-node $F_{13}$ at time $t = 6$ in the lemniscate depicted on the left hand side.

## 2.4 Backtrack Cost Trees

For analyzing the behavior of Max-sum on factor graphs with an arbitrary (finite) number of cycles, Zivan *et al.* proposed the use of a *backtrack cost tree* (BCT) [28]. It allows one to trace, for each belief, the entries in the cost tables held by function-nodes that were used to compose this belief. That is, what were the components of the assignment's cost. Their analysis included insights regarding the constructions of beliefs from costs incurred by constraints. Thus, for every pair of constrained variables, $X_i$ and $X_j$, for each $x \in D_i$, $x' \in D_j$, the cost incurred by the constraint for assigning $x$ to $X_i$ and $x'$ to $X_j$ was denoted as $R(X_i = x, X_j = x')$. Formally, a BCT is definde as follows:

▶ **Definition 1.** *A Backtracking Cost Tree (BCT) is defined for a belief that appears either in a message sent from variable $X_i$ at time $t$, to a function node connecting it to a variable $X_j$ or to a message sent from that function node to variable $X_i$. The belief is regarding the cost of assigning some $x \in D_i$ to $X_i$. Without loss of generality, we will elaborate on the first among these two and denote it as $BCT^t_{i=x\to j}$.*

*The belief, as constructed by the Max-sum algorithm, is a sum of various components, and the tree is composed from them. At the root is the belief, i.e., a cost for assigning some $x \in D_i$ to $X_i$, and it is connected to all nodes it received a message from at time $t-1$, with the edges containing the beliefs it was passed that ended up in the calculation of the belief it sent. Each of those nodes is connected itself to the nodes that send it messages at time $t-2$, with the edges containing the beliefs that passed to it that ended up in its message. The tree leaves are all at time* 0 *(see Figure 2 (b)).*

For a single-cycle factor graph, the BCT for every belief is a chain. Factor graphs with multiple cycles include variable-nodes with more than two neighbors, and thus, the BCTs of their beliefs include nodes with multiple children.

A BCT starts from the end point (i.e., the root of the BCT as presented in Figure 2 (b)), which is the *belief* (cost) of assigning to $X_i$ some value $x$ from its domain $D_i$, as sent to a neighboring node.

The values from which that belief was calculated can then be backtracked to the messages and costs due to all the individual constraints that were summed up to create that belief. An example of such a tree for a belief generated when Max-sum solves the factor-graph depicted in Figure 2(a) is depicted in Figure 2(b).

For each BCT, there is an implied assignment tree that consists of the value assignments that the variables at each time-point of the tree would need to be assigned in order to incur the costs included in the BCT. The value assignment selected by a variable at time $t$ is the one with the minimal sum of beliefs sent to the corresponding variable-node at iteration $t-1$. The tree for this minimal sum of beliefs will be denoted by $BCT_i^t$, as it does not depend on any specific belief that appears in a message to another variable.

## 2.5 Convergence Properties

Belief propagation converges in linear time to an optimal solution when the problem's corresponding factor graph is acyclic [16]. For a single-cycle factor graph, we know that if belief propagation converges, then it is to an optimal solution [8, 24]. Moreover, when the algorithm does not converge, it periodically changes its set of assignments. In order to explain this behavior, Forney *et al.* show the similarity of the performance of the algorithm on a cycle to its performance on a chain, whose nodes are similar to the nodes in the cycle, but whose length is equal to the number of iterations performed by the algorithm. One can consider a sequence of messages starting at the first node of the chain and heading towards its other end. Each message carries beliefs accumulated from costs added by function-nodes. Each function-node adds a cost to each belief, which is the constraint value of a pair of value assignments to its neighboring variable-nodes. Each such sequence of cost accumulation (route) must at some point become periodic, and the minimal belief would be generated by the minimal periodic route. If this periodic route is consistent (i.e., the set of assignments implied by the costs contain a single value assignment for each variable), then the algorithm converges. Otherwise, it does not [8].

Recently, these insights were generalized such that similar statements can be made when the algorithm is solving factor graphs with multiple cycles. Specifically (using BCTs), Zivan *et al.* proved that, as in the single cycle case, on every finite factor graph, Max-sum at some point in time starts to repeatedly follow a path that minimizes its beliefs. When a large enough damping factor is used, this minimal path is indeed a minimal path in the factor graph, and thus, if it is consistent, then the algorithm converges to an optimal solution [28].

# 3 The Effect of Asynchronous Execution

In order to analyze the differences in the performance of Syn_Max-sum and Asy_Max-sum, one must investigate the differences in the structure of the BCTs of beliefs sent by the algorithms' nodes. In Syn_Max-sum, the height of a BCT for a belief included in a message sent at iteration $t$ is $t$ and, for each node in the tree, the heights of the sub-trees rooted by each of its children nodes are equal. On the other hand, in Asy_Max-sum, messages can have different delays and, thus, each sub-tree in a BCT can have a different height.

Our first theoretical property addresses the results proved in [28] regarding the convergence of the synchronous version of Max-sum (Syn_Max-sum). More specifically, we prove that the property that was proved in Lemma 1 in [28], and was used to prove the main theorem of this study (i.e., the main theorem in [28]), is not guaranteed when the algorithm is performed asynchronously in an environment that includes message latency.

▶ **Proposition 1.** *In the presence of message delays, unlike Syn_Max-sum, Asy_Max-sum is not guaranteed to converge to a minimal repeated route.*

**Proof:** The structure of the BCTs of the beliefs that are exchanged by agents, depend on the timing of the arrival of messages from which they are composed. Each BCT (and as a result, the corresponding belief that it demonstrates its construction), is an outcome of a specific combination of message delays, resulting in different orders of message arrivals and the number of such combinations is exponential in the maximal number of messages that the beliefs they carry can be included in the BCT. Moreover, the combination of message delays that resulted in a specific minimal route of beliefs is not guaranteed to repeat itself. Thus, even if the algorithm reaches a minimal route, it may not repeat it. □

The proposition above seems to put an end to the natural wish that the convergence property of Syn_Max-sum can be established for Asy_Max-sum as well. However, the differences between the executions of the two versions of the algorithm can be minimized. More specifically, the effect caused by sub-trees of the BCTs having different heights in Asy_Max-sum can be significantly reduced through the use of damping.

Denote by $layer_k$ the set of nodes of a BCT with depth $k$ (distance from the root), and by $BCT_k$ the layers of the BCT with depth $k$ or less. We will say that a $layer_k$ is *effective* if and only if there exists a belief calculated using $BCT_k$ that is different than the belief calculated when taking into consideration the complete BCT. For each BCT $B$, we say that its effective BCT $B'$ is $BCT_{k'}$ such that $layer_{k'}$ is effective and for any $layer_k$ that is effective in $B$, $k' \geq k$.

▶ **Lemma 1.** *When asynchronous DMS (Asy_DMS) is performed with a large enough damping factor*[2]*, in an environment including bounded message delays, there exists a finite number of non-concurrent steps*[3] *of the algorithm $ns_1$, such that in the steps following it, for every two beliefs included in the same message, if $layer_k$ in each of the corresponding BCTs is effective, then the number of nodes in $layer_k$ of both BCTs are equal.*

**Proof:** Since delays are bounded, there exists a number of non-concurrent steps $ns_0 < ns_1$ in which the roots of the BCTs of all beliefs received in messages for every step following $ns_0$ have the same number of children. This will be true for all non-concurrent steps $ns > ns_0$ and, thus, layers of BCTs of beliefs that are sent in the same message with depth $k$ following $ns \geq ns_0 + \delta k$ (where $\delta$ is the maximal size of a message delay, in terms of non-concurrent steps) must have the same number of nodes. Damping with a large enough damping factor, causes the bottom layers of BCTs to have less influence on the calculation made by the nodes in the algorithm following each computation step (see [28] for details). Let $\epsilon$ denote the smallest cost that can affect the nodes' actions in the algorithm. If we wait for a sufficiently large enough number of steps, the maximal sum of costs in the BCTs, of steps performed before $ns_0$ will be smaller than $\epsilon$. We use $ns_1$ to denote that sufficiently large enough number of steps. □

An immediate corollary from Lemma 1 is that in Asy_DMS (which is using a large enough damping factor), following $ns_1$, the effective BCTs of all beliefs included in each message have the same number of nodes. This reduces the possible differences between beliefs that can be generated by each node. Moreover, for the case that the algorithm does converge, the effect of the asynchronous performance vanishes, as we prove below.

[2] For an analysis of the size of the damping factor required, with respect to the largest number of neighbors (degree) that a node in the factor graph has, see [28].

[3] We consider a step to be an action that starts when a node in the graph received some messages (at least one), performed computation and ends when it sent some messages (at least one).

▶ **Proposition 2.** *When Asy_DMS using a large enough damping factor, is performed in an environment with bounded message delays, if after performing $ns_2 > ns_1$ ($ns_1$ as described in Lemma 1) non-concurrent steps, it reaches a minimal consistent route (i.e., all nodes perform $k$ sequential asynchronous steps in which the value assignments corresponding to the minimal route are selected), then it will repeatedly follow this route (i.e., it has converged).*

**Proof:** As established above, following $ns_1$, the effective BCTs for beliefs included in the same message have the same number of nodes (in each layer and altogether) regardless of message delays. When the algorithm reaches a minimal consistent route, the beliefs corresponding to this minimal route involve only one value in each domain, and the belief corresponding to it is minimal in each message. Additional nodes added to the BCTs of the beliefs corresponding to the assignments in the minimal route represent costs in the entries of the cost tables of function-nodes that are part of the minimal route. Hence, they will not change its minimal property or the choice of the minimal route assignments, i.e., for every $ns > ns_2$ the effective $BCT_i^{ns}$ will be identical. Similarly, the addition of nodes to BCTs of beliefs corresponding to assignments that are not included in the minimal route represent costs that belong to routes with larger overall costs. ☐

Proposition 2 has a major importance to our discussion. Both the asynchronous and the synchronous versions of DMS will converge when they reach a consistent minimal path (i.e., the differences between them can exist only when the minimal path is inconsistent. In such a case, the synchronous execution version will repeat the minimal non consistent route while the asynchronous execution version may leave it and explore other routes).

## 4 Experimental Evaluation

In order to evaluate the implications of asynchronous execution (compared to synchronous execution) and message latency on the different versions of Max-sum, we used an asynchronous simulator, in which agents are implemented by Java threads. It includes a *mailing agent* that simulates the delays of messages as suggested by [29]. Using this type of simulator allows us to implement any type of message delay pattern. Other simulators, such as ns-3 [12, 1], offer a number of communication patterns from which one can select. However, we prefer the use of the simulator proposed in [29], which allows complete flexibility in the design of the message delay pattern and it allows to measure run-time in implementation independent units. Thus, the results are presented as a function of the number of *non-concurrent logic operations* (NCLOs). The atomic logic operations in these algorithms are the evaluation of the cost of a combination of two assignments (i.e., an access to the cost table of a function-node). Each agent performed the computation for the function-nodes that were assigned to it. We used a greedy heuristic to evenly assign function-nodes to agents and, thus, increase concurrency. In order to simulate message delays, for each message sent between nodes that their roles were performed by different agents, a delay in terms of NCLOs was selected, and the message was delivered to the receiving agent after that agent had the opportunity to perform this number of logic operations.

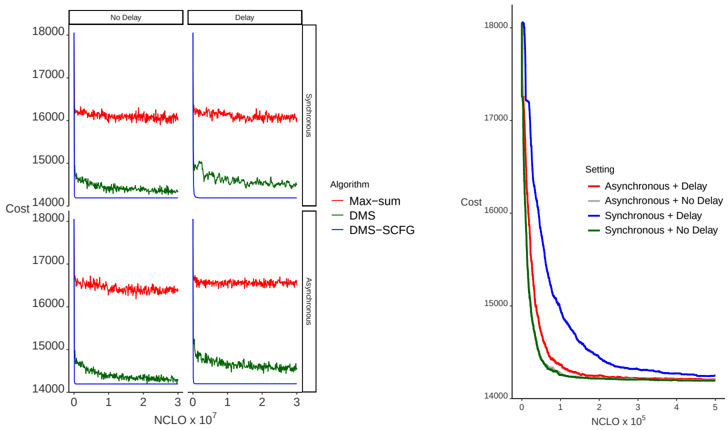
We evaluated the algorithms on problems including $50$ agents, which are too large for complete DCOP algorithms to solve. These included random graph problems, graph coloring problems, scale-free network problems, and overlapped solar systems problems (details below).

In each experiment, we randomly generated 50 different problem instances. The results presented in the graphs are an average of those 50 runs. In order to demonstrate the convergence of the algorithms, we present the sum of costs of the constraints involved in the assignment that would have been selected by each algorithm every $100K$ NCLOs. We also performed *t-tests* to evaluate the significance of differences between all presented results.

As mentioned above, the experiments were performed on four types of distributed constraint optimization problems. Each type of problem exhibits a different level of structure in the constraint graph topology and in the constraint functions. All problems were formulated as minimization problems.

- **Random Graph Problems:** These problems are random constraint graph topologies with density $p_1 = \{0.1, 0.6\}$. They include variables with 10 values in each domain. The cost tables held by function-nodes include costs that were selected uniformly between 100 and 200. Both the constraint graph and the constraint functions are unstructured.
- **Graph Coloring Problems:** These problems are random constraint graph topologies in which each variable has three values (i.e., colors), and all constraints are "not-equal" cost functions, where an equal assignment of neighbors in the graph incurs a random cost between 100 and 200 and non equal value assignments incur zero cost. Such random graph coloring problems are commonly used in DCOP formulations of resource allocation problems. We set the density to $p_1 = 0.05$ and had three values (i.e., colors) in each domain [27, 6, 4].
- **Scale-free Network Problems:** Problems generated using the model by [2]. An initial set of 10 agents was randomly selected and connected. Additional agents were added sequentially and connected to 3 other agents with a probability proportional to the number of links that the existing agents already had. The cost of each joint assignment between constrained variables was independently drawn from the discrete uniform distribution from 100 to 199. Each variable had 10 values in its domain. Similar problems were previously used to evaluate DCOP algorithms by Kiekintveld *et al.* [9]. The constraint graph is somewhat structured but the constraint functions are unstructured.
- **Overlapped Solar Systems Problems:** The overlapped solar system is a realistic problem, inspired by the Constant Speed Propagation Delay Model implemented in the ns-3 simulator [12, 1]. The graph topology is inspired by scale-free networks. An initial set of 5 agents are randomly selected to be the centers of the solar systems, and they are connected. Each of these agents $A_i^c$ is assigned two coordinates that are drawn from a continuous uniform distribution: $x_i^c \sim U(0,1)$ and $y_i^c \sim U(0,1)$. All other agents (i.e., stars in the solar systems) are randomly assigned to one of the solar systems. The index $c$ represents the solar system in which the agent is assigned too, and it is equal to the index of the center agent of the solar system (i.e., if $A_i^c$ is the center of a solar system, then $i = c$). The coordinates for an assigned agent ($A_j^c$ where $j \neq c$) are drawn from a Normal distribution as follows: $x_j^c \sim N(\mu = x_i^c, \sigma = 0.05)$ and $y_j^c \sim N(\mu = y_i^c, \sigma = 0.05)$ based on the location of the center of the solar system that it was attached to.
  The probability that two arbitrary agents $A_i$ and $A_j$ will be neighbors is defined by $p_{ij} = (1 - \frac{distance_{ij}}{maxDistance})^\beta$ where $distance_{ij}$ is the Euclidean distance between agents $A_i$ and $A_j$, $maxDistance$ is the Euclidean distance between agent $A_i$ to the farthest agent, and $\beta$ expresses the changes in the probability that both agents will be neighbors as a function of their distance (in our experiments we used $\beta = 3$). For each pair agents, a random probability $p_r \in [0,1]$ was generated, and two agents are considered as neighbors if $p_r < p_{ij}$. Costs between connected agents were selected uniformly between 100 and 200.
  While the structure of these problems is similar to scale-free networks, the addition of the geographic locations of nodes allows one to calculate the size of message delays with respect to physical distance as specified below.

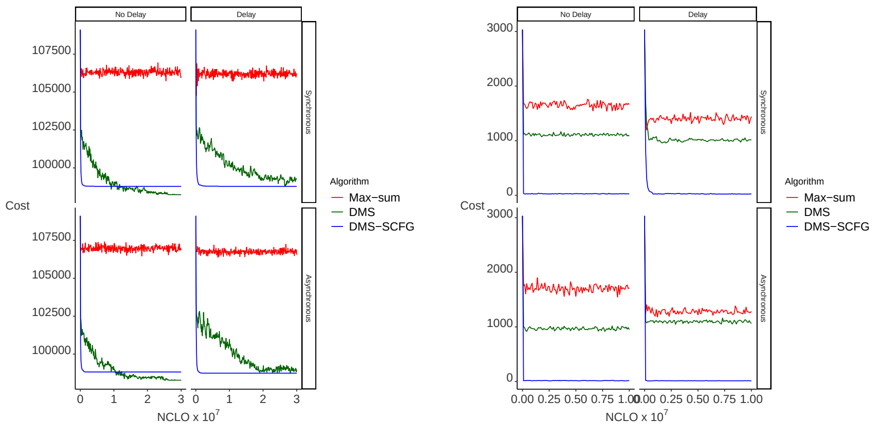
For random uniform problems, graph coloring problems, and scale-free network problems, all algorithms were run in a setup with no message delays and a setup with random message delays selected uniformly from the range $(0, 10K)$ NCLOs. For overlapped solar systems problems, in addition to the no message delay setup, the delay for each sent message between agents $A_i$ and $A_j$




(a) (b)

**Figure 3** (a) Solution quality as a function of $NCLOs$, of Max-sum versions solving sparse random problems ($p_1 = 0.1$). (b) A closer look at the solution quality of DMS-SCFG versions on these problems.

was drawn from a Poisson distribution $Poisson(\Gamma \cdot distance_{ij})$ NCLOs where $\Gamma$ is the average delay. This is in contrast to the Constant Speed Propagation Delay Model implemented in ns-3 where the delays that were calculated as a function of the distance between the geographic location of the nodes in the communication graph, were fixed and not sampled [12, 1].

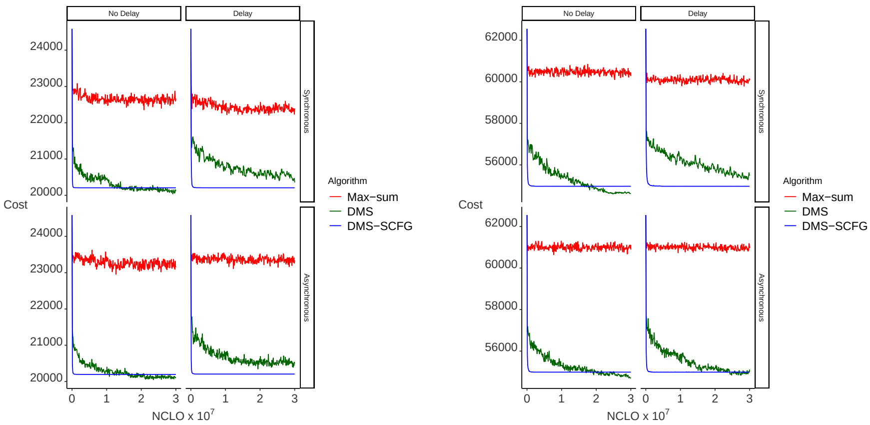



(a) (b)

**Figure 4** Solution quality as a function of $NCLOs$, of Max-sum versions solving dense random problems ($p = 0.6$) (a) and graph coloring problems (b)).

## 4.1 Results

Figure 3(a) presents the quality of solutions produced by the different versions of Max-sum when solving sparse random graph problems with $p_1 = 0.1$. Each figure presented in this sections includes four graphs, presenting results of the algorithms when performing synchronously, asynchronously,




(a) (b)

**Figure 5** Solution quality as a function of $NCLOs$, of Max-sum versions solving scale-free network problems (a) and overlapped solar systems problems (b)).

with message delays and without. The versions include Max-sum, DMS with $\lambda = 0.9$, DMS-SCFG.[4] Asy_Max-sum (with and without message delays) traversed solutions with higher costs on average than Syn_Max-sum. The results of the different runs of the algorithms were scattered and, thus, the differences from the synchronous versions were not found to be statistically significant. Asy_DMS, on the other hand, performed similarly to Syn_DMS, with and without message delays (as expected following Proposition 1).

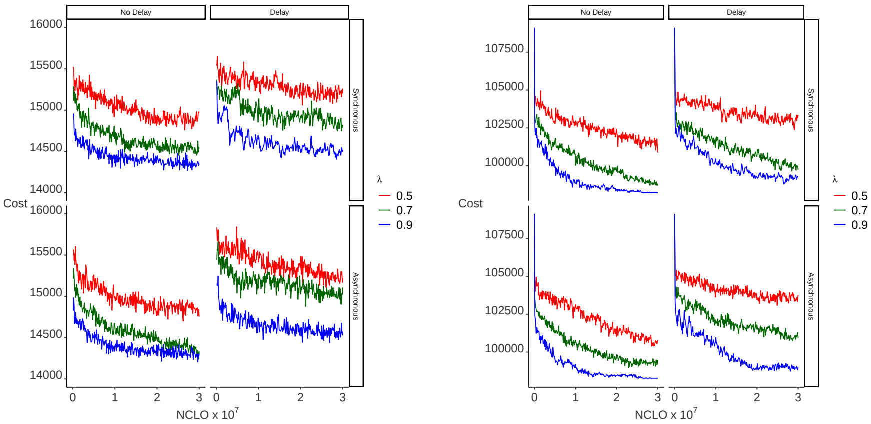
Another observation is that all versions of DMS-SCFG converged very fast compared to the other versions of the algorithm. Figure 3(b) provides a closer look that allows one to better compare their convergence rates. Both the synchronous and the asynchronous versions converge at the same rate in environments that do not include message delays. Clearly, message delays affect the synchronous version more than the asynchronous version of the algorithm. Nevertheless, in all execution modes, the algorithm converges very fast to solutions with the same quality.

Figure 4(a) presents the results for the same algorithms solving dense random graph problems with $p_1 = 0.6$. While the results seem similar to the results presented in Figure 3(a), there are fewer differences between the Max-sum versions. On the other hand, on these problems, the DMS versions in scenarios that do not include message delays find high quality solutions faster and converge.

Figure 4(b) presents the results of the algorithms solving graph coloring problems. It is apparent that the exploration performed by Max-sum and DMS is less effective on these problems, and thus, the advantage of DMS-SCFG is prominent. Moreover, in the presence of message delays, standard Max-sum improves its performance. We assume that delays break the very structured execution on this type of problems, and has a positive exploration affect. This affect is diminished when damping for the same properties that we established in the section titled "The Effect of Asynchronous Execution."

The results of the algorithms when solving scale free network and the overlapping solar system problem are presented in in Figure 5. They were found to be similar to the results presented in Figure 4(a) for the dense random problems. The differences in the performance of Asy_Max-sum from Syn_Max-sum was found to be significant when solving scale-free networks, with and without message delays. No significant difference was found between the synchronous and asynchronous

[4] DMS-SCFG is the damped Max-sum (DMS) algorithm with split constraint factor graphs (SCFGs). We used the 0.4-0.6 version of DMS-SCFG, which was found to perform best by [4].




(a) (b)

**Figure 6** Solution quality as a function of $NCLOs$, of DMS with different $\lambda$ values, solving random uniform problems with $p_1 = 0.1$ (a) and $p_1 = 0.6$ (b)).

versions when solving overlapped solar system problems. It seems for these problems that the similar structure has a more major effect on the behavior of the algorithms than the pattern of the message delays.

In our second set of experiments we evaluated the influence of the selection of the damping factor on the effect that asynchronous execution and message latency have on DMS's performance. Figure 6 presents the results of the algorithm with three different values of the damping parameter, i.e., $\lambda = 0.5$, $\lambda = 0.7$ and $\lambda = 0.9$, solving sparse (a) and dense (b) random uniform problems. As expected from the properties established in Propositions 1 and 2, asynchronous execution affects the performance of all versions of DMS when it does not converge. However, it is apparent that the $\lambda = 0.9$ version is less affected by message delays in the asynchronous execution, as expected. Similar results were obtained for all types of problems and were omitted to avoid redundancy.

In order to compare the effect that message delays have on the agents performing synchronously and asynchronously, we measured the average number of NCLOs in which agents were idle in each mode of execution of the algorithm. The results are presented in Figure 7. It includes for each algorithm, in each mode of execution, the average ratio of the number of NCLOs in which the agent was idle (i.e., waiting for message to arrive) and the total number of NCLOs the algorithm was executed. It is apparent that when solving all problem types, the agents performing asynchronously spend less time idle than the agents performing synchronously. This difference between the performance of the synchronous and the asynchronous versions was most apparent in DMS_SCFG. Nevertheless, while the difference in the time the agents spent idle when performing this type of the Max-sum algorithm, the synchronous and the asynchronous versions were most similar in their convergence time and the solution quality.

## 4.2 Discussion

The advantage of DMS over standard Max-sum, when solving graphs with multiple cycles, was reported empirically in a number of studies (e.g., [4]) and explained theoretically by [28]. In Max-sum, costs that are aggregated in the beginning of the run are duplicated in every node of the graph that has more than two neighbors and, thus, they are taken into consideration an exponential number of times in the calculation of beliefs and in the assignment selection. Damping reduces the weight of these costs in the belief calculation until it becomes negligible. A similar phenomenon reduces the

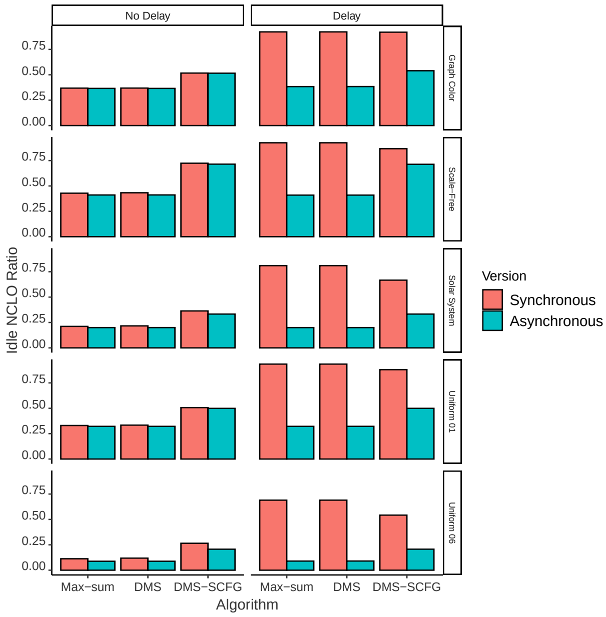



**Figure 7** Ratio between the number of $NCLOs$ in which the agents were idle and the total number of $NCLOs$ for all algorithms and all execution modes.

differences between the performance of Syn_DMS and Asy_DMS. As we established in the corollary of Lemma 1, when using a large enough damping factor, the effect of BCTs with different heights is eliminated in DMS and, thus, after enough NCLOs are performed, the effective BCTs of the beliefs in each message have the same number of nodes. The results comparing DMS with different damping factor values, demonstrate the need to use a high damping factor in order to achieve robustness to message delays. This empirical evidence, strengthens the property established in Lemma 1 and its corollary, that if the damping factor used in not high enough, the effect of the lower layers of the BCTs, which may have different structure and a different number of nodes, on the generation of beliefs by the nodes, is not eliminated. Thus, message delays have a greater effect on the algorithm's performance when the damping factor used is not low. Finally, Asy_DMS-SCFG maintains the fast convergence properties and the quality of the solutions of the synchronous version. It is also robust to message latency.

# 5 Conclusions

In this paper, we filled the gap in the Max-sum literature on the difference of synchronous and asynchronous executions of the algorithm in distributed environments. Our theoretical analyses revealed that, unlike its synchronous counterpart, the asynchronous version of Max-sum in the presence of message latency can cause the propagation of inconsistent beliefs, resulting in the loss of guaranteed properties (Proposition 1). However, not all is lost as one can use damping to minimize this effect and, subsequently, ensure that when asynchronous DMS finds a minimal route, it will converge, as does the synchronous version (Proposition 2). Finally, experimental results show that when the algorithm is further optimized through split constraint factor graphs, it converges very fast to high-quality solutions even in the presence of message delays. Taken together, these results extend significantly our understanding of Max-sum in distributed environments with more realistic messaging assumptions, propose algorithmic tools that are theoretically grounded to alleviate the issues raised, and enable a more effective use of Max-sum by real-world practitioners.

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