An Example of Non-Existence of Riley Equilibrium in Markets with Adverse Selection

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Rothschild and Stiglitz [1976] proposed a model of a competitive market with adverse selection and showed that a (pure strategy) Nash equilibrium may not exist. Among the solutions proposed to deal with this problem, a particularly influential one is the notion of Riley (or reactive) equilibrium [Riley, 1979]. We give an example that shows that, when consumers are not ordered along a single dimension of heterogeneity, a Riley equilibrium may also not exist.

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Rothschild and Stiglitz [1976] proposed a model of a competitive market with adverse selection where firms compete by offering insurance contracts to consumers. They showed that a (pure strategy) Nash equilibrium may not exist. Among the solutions proposed to deal with this problem, a particularly influential one is the notion of reactive equilibrium [Riley, 1979] – also known as Riley equilibrium. In a Riley equilibrium, before introducing a new contract, each firm anticipates how other firms would react to it. If, in response to the new contract, another firm would introduce a profitable contract that makes the original firm lose money, it chooses not to introduce it. Additionally, this new profitable contract cannot be made unprofitable by the introduction of another contract.

Riley [1979] showed that the unique reactive equilibrium in the model of Rothschild and Stiglitz is the “least costly separating equilibrium.” Engers and Fernandez [1987] generalized Riley’s result, while maintaining the assumption of one-dimensional, ordered types. However, subsequent empirical and theoretical work has shown that consumers are often heterogeneous in more than one dimension, and that this is important for various policy questions.

In this note, we show that a Riley equilibrium may not exist. We give a natural example of a market with no Riley equilibria. The set of contracts is simple, as firms offer only two contracts, but consumer types are not ordered along a single dimension, unlike the case considered by Riley and Engers and Fernandez. This suggests that, to obtain predictions in more realistic environments, other solution concepts are necessary.

1 Preliminaries

We start with the formal definition of a Riley equilibrium based on Engers and Fernandez’s [1987] generalization of Riley [1979]. The model studies the interaction between uninformed firms who sell different contracts to privately-informed consumers. Firms compete by offering menus of contracts to consumers. Let \( X = \{x_1, \ldots, x_C\} \) be the finite set of possible contracts. A price vector \( \mathbf{p} = \{p_1, \ldots, p_C\} \) specifies a price for each contract.

Consider a situation where firms offer price vectors \( \mathbf{p}_1, \ldots, \mathbf{p}_N \), and let \( \mathbf{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_N\} \) be the set of price vectors being offered. In order to determine whether this is a Riley equilibrium, we must consider what happens when an entrant offers a new price vector \( \mathbf{p}_0 \).

1 As Rochet and Stole [2003] point out, there is a close relationship between models with multidimensional types and those with unordered types. For example, any model with finitely many multidimensional types can be written as a model with one-dimensional types. However, the resulting one-dimension type space is typically not ordered.

2 See, for example, Finkelstein and McGarry [2006], Cohen and Einav [2007], Fang et al. [2008]. Multidimensional heterogeneity is discussed in detail by Chiappori and Salanié [2000], Smart [2000], Heckman [2004], Veiga and Weyl [2014], Araujo and Moreira [2010], Azevedo and Gottlieb [2014].

3 See also Handel et al. [2015].
We write $p^0 \oplus P$ to denote the new set of price vectors that is obtained by including vector $p^0$ in $P$, i.e., $p^0 \oplus P = \{p^0, p^1, ..., p^N\}$. For any $p \in P$, let $\Pi(p, P)$ denote the per-unit profit obtained by a firm that offers price vector $p$ when the set of all offered price vectors corresponds to $P$. In applications, $\Pi$ is obtained by specifying preferences and the private information of informed consumers (as we will do in Section 2).

**Definition 1.** A Riley equilibrium is a set of price vectors $P$ with the following properties:

- Firms do not incur losses: $\Pi(p, P) \geq 0$ for all $p \in P$.
- For any new price vector $p_A \notin P$ that gives a positive profit
  
  \[ \Pi(p_A, p_A \oplus P) > 0, \]

  there exists another price vector $p_B \notin P$ such that:
  - $p_A$ incurs losses when $p_B$ is included:
    \[ \Pi(p_A, p_B \oplus p_A \oplus P) < 0, \]
  - $p_B$ does not incur losses when any new price vector $p_C$ is included ($p_B$ is “safe”):
    \[ \Pi(p_B, p_C \oplus p_B \oplus p_A \oplus P) \geq 0. \]

Definition 1 states that a set of price vectors is a Riley equilibrium if for any new contract that would generate a profit to some firm $A$, there exists an “entrant” $B$ that would make $A$’s contract unprofitable. Moreover, this entry by firm $B$ is “safe” in the sense that there is no additional entry by another firm that would make firm $B$ lose money. Notice that when there are no safe prices, Riley equilibria coincide with (pure strategy) Nash equilibria.

## 2 An Example of non-Existence of Riley Equilibria

We now show that Riley equilibria may not exist when types are not ordered. Our example is based on a variation of the Hotelling model. Consumers are uniformly distributed in the unit interval $\theta \in [0, 1]$. There are two possible contracts: $X = \{0, 1\}$. A type-$\theta$ consumer who buys contract $x$ at price $p$ gets utility

\[ K - \frac{(x - \theta)^2}{\alpha} - p, \]
where $\alpha > 0$ parameterizes the similarity between contracts (it is the inverse of the transportation cost in the standard Hoteling model) and $K$ is large enough to ensure that the market is served. Selling contract $x \in \{0, 1\}$ to type $\theta$ costs

$$\left(\theta - \frac{1}{2}\right)^2.$$

Notice that the types in the extreme points have the strongest preference for one of the contracts and are the most costly to serve. Types in the middle have the weakest preference between contracts and are the cheapest consumers.

The type who is indifferent between both contracts $\theta^*$ is determined by:

$$K - \frac{\theta^2}{\alpha} - p_0 = K - \frac{(1 - \theta)^2}{\alpha} - p_1 \implies \theta^* = \frac{1 - \alpha (p_0 - p_1)}{2}.$$

Then, because types are uniformly distributed, the demand for contract 0 is:

$$D_0(p_0, p_1) = \begin{cases} 
0 & \text{if } 1 + \alpha (p_1 - p_0) < 0 \\
\frac{1 + \alpha (p_1 - p_0)}{2} & \text{if } 0 \leq 1 + \alpha (p_1 - p_0) \leq 2 \\
1 & \text{if } 1 + \alpha (p_1 - p_0) > 2 \end{cases}.$$

By symmetry, the demand for contract 1 is analogous. We say that a price vector $(p_0, p_1)$ yields an interior allocation if $0 < D(p_0, p_1) < 1$.

Because both contracts cost the same and willingness to pay is symmetrically distributed, we can define $C(q)$ as the expected per-unit cost of selling to the mass $q$ of consumers with the highest willingness to pay for each contract:

$$C(q) = \int_0^q \left(\frac{\theta - \frac{1}{2}}{q}\right)^2 d\theta = \frac{q^2}{3} - \frac{q}{2} + \frac{1}{4}.$$

For interior allocations, the per-unit profit of selling $x = i$ at price $p_i$ equals

$$p_i - C(D(p_i, p_{-i})) = p_i - \frac{1 - \alpha (p_{-i} - p_i) + \alpha^2 (p_{-i} - p_i)^2}{12}. \quad (1)$$

The per-unit profit function $\Pi$ can be easily calculated from equation (1).\footnote{To do so, let $p_i^* (P) \equiv \min\{\tilde{p}_0 : (\tilde{p}_0, \tilde{p}_1) \in P\}$ denote the lowest price being charged for contract $i \in \{0, 1\}$. The per-unit profit of selling contract $i$ is

$$\phi_i(p_i, P) \equiv \begin{cases} 
0 & \text{if } p_i \neq p_i^* (P) \text{ or } p_i^* (P) - p^{*,-i}_i (P) \geq \frac{1}{\alpha} \\
p_i - \frac{1 - \alpha (p_{-i} - p_i) + \alpha^2 (p_{-i} - p_i)^2}{12} & \text{if } p_i = p_i^* (P) \text{ and } |p_i^* (P) - p^{*,-i}_i (P)| < \frac{1}{\alpha} \\
p_i - \frac{1}{12} & \text{if } p_i = p_i^* (P) \text{ and } p^{*,-i}_i (P) - p_i^* (P) \geq \frac{1}{\alpha} \end{cases}.$$}

The first line says that the firm gets zero profits if some other firm is charging a higher price, or if no consumer
It is helpful to consider the existence of Nash equilibria before turning to Riley equilibria. Lemma 1 shows that Nash equilibria do not exist when horizontal differentiation between contracts is low (i.e., $\alpha$ is high):

**Lemma 1.** Let $\alpha > 12$. Then, no (pure strategy) Nash equilibrium exists.

**Proof.** In any (pure strategy) Nash equilibrium, profits on all contracts are zero. Let $(p_0, p_1)$ denote the lowest prices charged for each contract. With interior allocations, setting per-unit profits (1) to zero for both contracts, yields $p_0 = p_1$ as long as $\alpha \neq 6$. Thus, there are three possible Nash equilibria: an interior, symmetric Nash equilibrium ($p_0 = p_1$) and two asymmetric Nash equilibria at the boundary (one where all consumers buy $x = 0$ and another where they all buy $x = 1$). In the symmetric candidate equilibrium, we must have

$$p_0 = p_1 = C\left(\frac{1}{2}\right) = \frac{1}{12}.$$  

In the asymmetric candidate equilibria, zero profits gives:

$$p_i = C(1) = \frac{1}{12},$$

where $i \in \{0, 1\}$ denotes the only contract being sold. Therefore, in all candidate equilibria, all traded contracts have price $\frac{1}{12}$.

We now verify that these are not Nash equilibria because firms can profit by offering a contract at a price slightly below $\frac{1}{12}$ and attracting a large mass of low-cost consumers. In the symmetric candidate equilibrium, the per-unit profit from selling a contract at price $p < \frac{1}{12}$ is

$$p - \frac{1 - \alpha \left(\frac{1}{12} - p\right) + \alpha^2 \left(\frac{1}{12} - p\right)^2}{12}.$$  

The derivative of this expression with respect to $p$ equals

$$1 - \frac{\alpha - 2\alpha^2 \left(\frac{1}{12} - p\right)}{12},$$

which, evaluated at $p = \frac{1}{12}$, becomes

$$1 - \frac{\alpha}{12} < 0 \iff \alpha > 12.$$  

is buying contract $i$. The second line specifies the per-unit profits if the allocation is interior, whereas the third line specifies profits when all consumers buy contract $i$. Total per-unit profits are the sum of per-unit profits from both contracts:

$$\Pi((p_1, p_2), P) \equiv \phi_0(p_0, P) + \phi_1(p_1, P).$$
Therefore, a small reduction in price below $\frac{1}{12}$ yields positive profits. Next, consider the asymmetric candidate equilibrium where only contract $x = i$ is sold at price $p_i = \frac{1}{12}$. By the previous calculations, selling the other contract $(x \neq i)$ at a price slightly below $\frac{1}{12}$ yields positive profits. Thus, no (pure strategy) Nash equilibrium exists when $\alpha > 12$. \hfill \Box

We now show that no prices are safe when $\alpha > 4$, so that the set of Riley equilibria coincides with the set of Nash equilibria.

**Lemma 2.** Let $\alpha > 4$. Then, a set of price vectors $P$ is a Riley equilibrium if and only if it is a (pure strategy) Nash equilibrium.

*Proof.* Fix a set of price vectors $p^A \oplus P$, and consider a new price vector $p^B$ that gives positive per-unit profits when added to $p^A \oplus P$. For this vector to yield positive profits, some consumers must prefer to buy a contract from the entrant offering $p^B$. Without loss of generality, let $x = 0$ denote this contract (the argument is symmetric for $x = 1$). Now, suppose a new entrant offers $p^C = (p^B_0 + 1, 0)$. With the addition of $p^C$, the firm offering $p^B$ obtains per-unit profits:

$$
\Pi(p^B, p^C \oplus p^B \oplus p^A \oplus P) = p^B_0 - \frac{1 + \alpha p^B_0 + \alpha^2 (p^B_0)^2}{12}.
$$

We claim that this expression is negative for any $p^B_0$. To see this, rewrite it as

$$
0 < \alpha^2 (p^B_0)^2 + (\alpha - 12) p^B_0 + 1,
$$

which is true because, when $\alpha > 4$, the expression on the RHS is a quadratic equation with no real roots. Therefore, no profitable price vector $p^B$ is safe and Riley equilibria coincide with (pure strategy) Nash equilibria. \hfill \Box

Combining Lemmas 1 and 2 gives our main result:

**Proposition 1.** No Riley equilibrium exists when $\alpha > 12$.

**References**


