On the geometric construction of a stabilizing time-invariant state feedback controller for the nonholonomic integrator

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Abstract

The paper presents a rather natural and elementary geometric construction of a stabilizing time-invariant state feedback law for the nonholonomic integrator. The key features of the particular construction are the direct inclusion of certain classical optimality considerations pertaining to the geodesics of the nonholonomic integrator, the confinement of all discontinuities in the feedback law to the z-axis, as well as a uniform exponential convergence result for the closed-loop system. The results of this treatment also have interesting implications for the control of nonholonomic systems in general, e.g., they highlight that for nonholonomic systems even the most natural seeming stabilizing feedback laws may not be amenable to a closed-form expression and may need to be formulated in more elaborate and implicit terms.

Key words: Nonholonomic systems; Nonlinear control; Feedback stabilization

1 Introduction

Ever since its introduction in the 1980s as a first example of a nonlinear control system that is fully controllable but for which no stabilizing continuous time-invariant state feedback exists (Brockett, 1983), the nonholonomic integrator (also often referred to as the Brockett integrator or the Heisenberg system, cf. the *Heisenberg group*) described by

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ xv - uy \end{pmatrix},$$

with states x, y, z, and inputs u, v, has prevailed as a prime example of nonlinear control that nicely encapsulates the intriguing peculiarities of general nonlinear control systems in a particularly simple and pure form. In the decades to follow, the nonholonomic integrator has served as a constant source of inspiration that spurred numerous further investigations and studies aimed at both gaining a better understanding of the fundamental limitations, as well as finding viable alternative strategies for its asymptotic stabilization in practice.

Given the nowadays well-known fundamental obstruction that no continuous time-invariant stabilizing state feedback exists for the nonholonomic integrator (Brockett, 1983), subsequent works have set themselves to finding viable solutions outside the class of continuous time-invariant state feedback, e.g., by considering discontinuous time-invariant feedback laws (Astolfi, 1994; Bloch and Drakunov, 1996; Astolfi, 1998; Hespanha and Morse, 1999; Bloch *et al.*, 2000; Liberzon, 2003; Dolgopolik and Fradkov, 2016) that all involve some explicit switching-based component, by considering time-varying state feedback (Morgansen and Brockett, 1999; Zuyev, 2016), or by its direct generalization of embedding the original system in a higher-dimensional space and then implementing time-invariant feedback using the extended state (Khaneja and Brockett, 1999).

In this paper, we revisit the classical problem of stabilizing the nonholonomic integrator by time-invariant state feedback. We first illuminate the problem in a very explicit geometric light, which although very elementary, appears to not have been given any considerations before. The simple insights gained from these geometric considerations very quickly point us towards what one may regard as a naturally occuring stabilizing feedback law for the nonholonomic integrator. We further build on the geometric insights by adding more analytical considerations into the picture, and establish a less intuitive *uniform exponential convergence* result that goes beyond the purely geometrical considerations.

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2 The nonholonomic integrator

In this section, we first show how the nonholonomic integrator can be quite naturally recovered from a simple geometric consideration of planar curves. This connection will provide a particularly clear description of the key steering mechanism of the nonholonomic integrator.

Given a planar C^1-curve $t \mapsto (x(t), y(t)) \in \mathbb{R}^2 \setminus \{0\}$, it is an interesting thought to attach to a point $(x(t), y(t))$ on the curve an angle via $\varphi(t) = \arctan\left(\frac{y(t)}{x(t)}\right)$, where we ignore the singularity for $x(t) = 0$ for a moment, cf. the illustration in Figure 1.

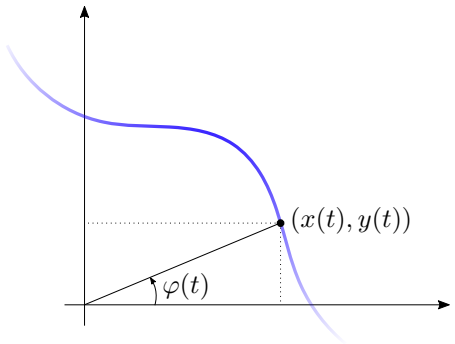

Fig. 1. Definition of the angle for a planar curve.

Formally differentiating φ with respect to time yields

$$\dot{\varphi} = \frac{1}{1 + (y/x)^2} \frac{d}{dt}\left(\frac{y}{x}\right) = \frac{x^2}{x^2 + y^2} \frac{x\dot{y} - \dot{x}y}{x^2} = \frac{x\dot{y} - \dot{x}y}{x^2 + y^2},$$

which is well-defined for all $(x, y) \in \mathbb{R}^2 \setminus \{0\}$. At this point, one could already additionally impose that the curve $t \mapsto (x(t), y(t))$ be produced by a single integrator

$$\dot{x} = u, \quad \dot{y} = v$$

to introduce external control inputs into the picture. This would then already provide us with an interesting and at the same time very manageable nonlinear control system with inputs to analyze.

But, instead, let us rearrange the equation for $\dot{\varphi}$ to

$$(x^2 + y^2)\dot{\varphi} = x\dot{y} - \dot{x}y \tag{1}$$

to eliminate the singularity. By the same line of argument as before, this now directly corresponds to the control system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ xv - uy \end{pmatrix},$$

which we recognize as the nonholonomic integrator.

While the influence of the control (u, v) to (x, y) is clear, our previous geometric considerations, specifically the relation (1), also shed light on the dynamics of the state z in terms of a certain area generated by the (x, y) curve, as illustrated in Figure 2.

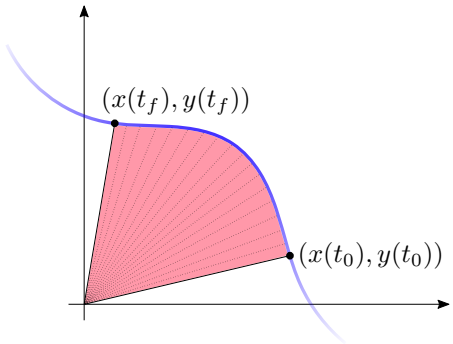

Fig. 2. The (signed) area A of the highlighted red sector formed by the curve segment $(x(t), y(t))_{t\in[t_0,t_f]}$ and the origin is $A = \frac{1}{2}\int_{t_0}^{t_f} x\dot{y} - \dot{x}y \,\mathrm{d}t = \frac{1}{2}(z(t_f) - z(t_0))$.

By comparing the dynamics of the nonholonomic integrator with the introductory geometric consideration, we can explain many known facts in an elementary way, as has already been done in numerous prior works, see, e.g., Brockett (1982); Vdovin *et al.* (2004). For instance, the frequently studied geodesics of the Heisenberg group (viewed as a sub-Riemannian manifold) can be readily interpreted in terms of an optimal control problem of minimizing the cost functional

$$J = \int_{t_0}^{t_f} \sqrt{u(t)^2 + v(t)^2}\,\mathrm{d}t = \int_{t_0}^{t_f} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\,\mathrm{d}t$$

subject to initial and terminal conditions given by (x_0, y_0, z_0) and (x_f, y_f, z_f), respectively. Using the geometric viewpoint, we can in turn interpret this optimal control problem as finding a planar curve $t \mapsto (x(t), y(t))$ in the xy-plane with minimal length that connects (x_0, y_0) and (x_f, y_f) with a specified area of the corresponding sector $A = \frac{1}{2}(z(t_f) - z(t_0))$. The planar curves connecting two points with a specified sector area and a minimal arc length are, of course, circular arcs (cf. the famous isoperimetric problem, see, e.g., Liberzon (2011)), except for when the sector area that needs to be generated is zero, where the planar curves are lines, which can, however, be regarded as arcs of circles with infinite radius.

Given the distinguished role of (generalized) circular arcs within the study of the Heisenberg group and related studies in nonlinear control theory Brockett (1982), it seems only natural to also give special consideration to the elementary geometric approach within the solution to the problem of stabilizing the nonholonomic integrator. This geometric formulation is examined in the next section.

3 Constructing the state feedback law from geometric considerations

Consider an initial state $(x, y, z) \in \mathbb{R}^3$, where for the moment, we exclude the cases of $(x, y) \neq (0, 0)$, as well as the cases of $z = 0$. Figure 3 describes a simple geometric illustration for solving the task of steering the nonholonomic integrator from the initial state to the origin by tracing out a circle segment with area $\frac{1}{2}|z|$. We note that when $(x, y) \neq (0, 0)$, the construction of that circle segment is unique.

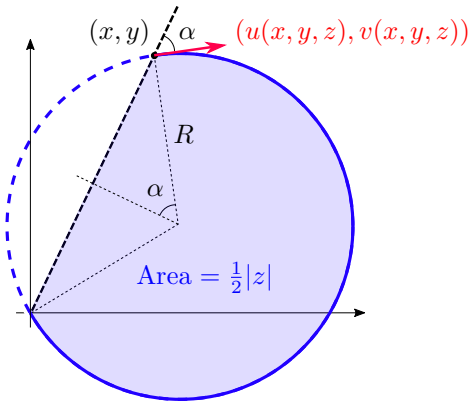

Fig. 3. Circle segment that connects (x, y) with $(0, 0)$ and has an area of $\frac{1}{2}|z|$. The red arrow indicates the instantaneous velocity vector (u, v) at the point (x, y). The figure also establishes an elementary geometric relation for the angle α between (u, v) and (x, y). In the depicted situation, we have $z > 0$, which is why the rotation is clockwise.

The shaded circle segment can be split up into a triangle and a circular sector. The area of the triangle is given by

$$A_{\text{triangle}} = R^2 \sin(\alpha) \cos(\alpha) = \frac{1}{2} R^2 \sin(2\alpha),$$

and the area of the circular sector is given by

$$A_{\text{sector}} = (\pi - \alpha) R^2.$$

The area of the shaded circle segment is thus given by

$$A_{\text{segment}} = \frac{1}{2}|z| = \frac{1}{2} R^2 (\sin(2\alpha) + 2(\pi - \alpha)), \tag{2}$$

where the radius R can be seen to satisfy

$$R = \frac{1}{2} \frac{\sqrt{x^2 + y^2}}{\sin(\alpha)}. \tag{3}$$

By plugging in (3) into (2), we obtain an analytical description for the relevant angle α as the solution to the transcendental equation

$$(x^2 + y^2)(\sin(2\alpha) + 2(\pi - \alpha)) = 4|z| \sin^2(\alpha). \tag{4}$$

For $(x^2 + y^2) \neq 0$ and $z \neq 0$, the transcendental equation (4) admits a unique solution within the open interval $(0, \pi)$. This solution cannot, however, be written in closed form as a function of $x^2 + y^2$ and $|z|$ due to the transcendental nature but that it is straightforward to compute the solution numerically. To see the existence of a unique solution of (4) for all $(x^2+y^2) \neq 0$ and $z \neq 0$, we can rewrite (4) as

$$\underbrace{\frac{\sin(2\alpha) + 2(\pi - \alpha)}{\sin^2(\alpha)}}_{=:\psi(\alpha)} = \frac{4|z|}{x^2 + y^2},$$

which also highlights that we can view α as a function of merely the *ratio* of $x^2 + y^2$ and $|z|$. A direct inspection shows that the function $\psi : (0, \pi) \to (0, \infty)$ defined above is bijective and further satisfies $\frac{d}{d\alpha}\psi < 0$ for all $\alpha \in (0, \pi)$ so that we may write $\alpha = \psi^{-1}(\frac{4|z|}{x^2+y^2})$, which is further smooth in (x, y, z) with $x^2+y^2 > 0$ and $|z| > 0$.

Figure 4 illustrates closed-loop solutions obtained from implementing the naturally associated preliminary feedback law

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \operatorname{sgn}(z) \sin(\alpha) \\ -\operatorname{sgn}(z) \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{pmatrix},$$

where α is the unique solution of (4) within $(0, \pi)$.

We note that depending on the initial angle of (x, y) in the xy-plane, the solutions stay on distinct "leaves" or "branches" and that these constitute a foliation of the three-dimensional state space. Along these lines, it is also interesting to observe that trajectories resulting from initial states with $\sqrt{x^2 + y^2} \gg |z|$ can be directly viewed as the end-pieces of trajectories originating from some corresponding initial states $(\tilde{x}, \tilde{y}, \tilde{z})$ with $\sqrt{\tilde{x}^2 + \tilde{y}^2} \ll 1$ but non-zero and $|\tilde{z}| \gg 1$, cf. Figure 4 (c).

By inspecting Figure 4 (a), the issue with points $(0, 0, z) \in \mathbb{R}^3$ also quickly becomes apparent. Besides the more superficial observation that the proposed feedback law is simply not well-defined in these cases, Figure 4 highlights very clearly that solutions starting at points (x, y, z) with $\sqrt{x^2 + y^2} \ll 1$ in the close vicinity of points $(0, 0, z)$ are branching out in all different directions depending on the initial angle of (x, y) in the xy-plane. This is also in accordance with the well-known fact that there are infinitely many geodesics of the Heisenberg group connecting the origin with any given point on the z-axis. As a result, it is impossible to find a continuous extension. A simple resolution is given by simply selecting one branch, say the dark blue one in Figure 4 (b), which will extend the feedback law to all points $(0, 0, z)$ with $z \in \mathbb{R}$, by setting $(u, v) = (1, 0)$.

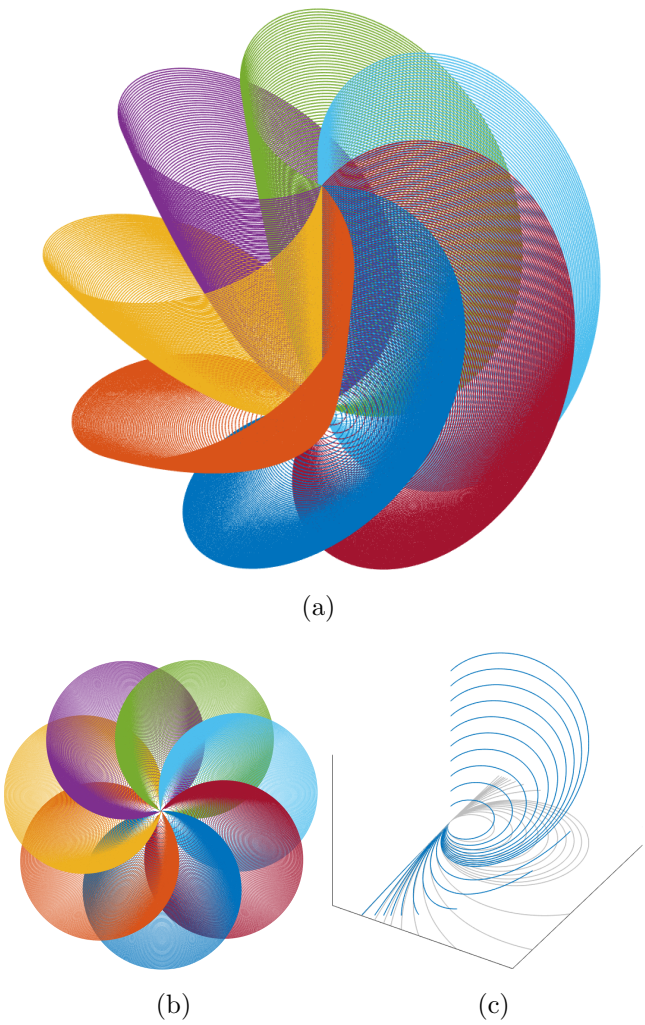

Fig. 4. (a) Closed-loop solutions obtained from the proposed feedback law and initial states (x, y, z) with $\sqrt{x^2+y^2} \ll 1$ but non-zero. (b) The top view of the upper plot verifies that the closed-loop trajectories are indeed tracing out circles as described in the geometric construction. (c) Several trajectories that are all associated with one particular leaf (in blue) along with their xy-projections (in grey).

The nature of the discontinuities on the z-axis as being due to the presence of infinitely many paths branching out for different $(0, 0, \epsilon)$ is reminiscent of other related examples highlighting fundamental topological obstructions for asymptotic stabilization with continuous time-invariant state feedback. One such classical example is given in terms of the inability to find a continuous vector field on the circle so that the south pole is asymptotically stable, see, e.g., Liberzon (2003). Moreover, the type of discontinuity, as well as the simple resolution via assigning a specific branch of the infinitely many can be considered very harmless when compared with discontinies that could produce non-differentiable solutions of the system or chattering.

Regarding the extension of the feedback law to the xy-plane, observe that as $|z| \to 0$, the solution of the transcendental equation approaches π, which when plugged into the feedback law initially only defined for $x^2+y^2 \neq 0$ and $z \neq 0$ yields

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}.$$

The main theorem of the paper builds on the preliminary considerations and establishes a suitable modification of the control law that yields a *uniform exponential decay* of the closed-loop trajectories. The preliminary feedback law proposed so far admits a constant magnitude of the input signal, i.e., $\|(u(t), v(t))\| \equiv 1$ and the challenge in establishing an exponential stability result is the determination of a suitable state-dependent gain.

Theorem 1 *For all points $(x, y, z) \in \mathbb{R}^3$ that satisfy both $x^2 + y^2 \neq 0$ and $z \neq 0$, define the feedback law*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\sqrt{x^2+y^2}+4\sqrt{|z|}}{\sqrt{x^2+y^2}} \begin{pmatrix} \cos(\alpha) & \operatorname{sgn}(z)\sin(\alpha) \\ -\operatorname{sgn}(z)\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where the occuring angle α is given as the unique solution of the transcendental equation

$$(x^2 + y^2)(\sin(2\alpha) + 2(\pi - \alpha)) = 4|z| \sin^2(\alpha)$$

in the interval $(0, \pi)$. The natural extension of this feedback law to the z-axis and the xy-plane via

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 4\sqrt{|z|} \\ 0 \end{pmatrix}, \quad \textit{and} \quad \begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix},$$

respectively, yields a globally stabilizing feedback law for

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u \\ v \\ xv - uy \end{pmatrix}$$

that is smooth almost everywhere except for discontinuities along the z-axis and non-differentiability on the xy-plane.

The closed-loop trajectories further exhibit uniform exponential convergence to the origin in the sense that with the positive definite and radially unbounded function

$$N(x, y, z) = \sqrt{x^2 + y^2} + 4\sqrt{|z|}$$

one has the (uniform) exponential decay

$$N(x(t), y(t), z(t)) \leq 3\, e^{-t}\, N(x(0), y(0), z(0)).$$

PROOF. The proof of the theorem can be split up into the two cases where $z(0) = 0$ and $z(0) \neq 0$.

Regarding the case $z(0) = 0$, we can observe that as $|z| \to 0$, the solution of the transcendental equation approaches π, which when plugged into the feedback law initially only defined for $x^2 + y^2 \neq 0$ and $z \neq 0$ yields

$$\begin{pmatrix} u \\ v \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix},$$

showing that the above definition yields a continuous extension of the original feedback law to the xy-plane. The feedback law clearly leaves the xy-plane invariant and results in an exponential convergence to the origin.

To study the case of $z(0) \neq 0$, we recall that by the construction of the feedback law from the elementary geometric consideration presented in the foregoing section, the closed-loop solutions will trace out boundaries of circle segments of area $\frac{1}{2}|z(0)|$ that connect $(x(0), y(0))$ with $(0, 0)$ in the xy-plane. In this non-degenerate case, we can express the quantities $\sqrt{x^2 + y^2}$ and $\sqrt{|z|}$ in terms of functions of the variable α. More specifically, by using the relations (2) and (3) from the geometric consideration, we end up with

$$\begin{aligned} N(x, y, z) &= \sqrt{x^2 + y^2} + 4\sqrt{|z|} \\ &= 2R\sin(\alpha) + 4R\sqrt{\sin(2\alpha) + 2(\pi - \alpha)}. \end{aligned}$$

Recall that this is also the gain that determines the magnitude of the (u, v) velocity vector in Figure 3 that in turn determines the angular speed

$$\omega(t) = \frac{N(x, y, z)}{R}$$

of $(x(t), y(t))$ about the center of the circle. This insight together with the linear relationship

$$\dot{\alpha}(t) = (1/2)\omega(t)$$

could be directly used to show that α converges to π exponentially, which means that in the xy-plane, the (x, y)-curve rotates into the origin in an exponential manner. However, to simplify arguments in the following, we will instead examine the dynamics of $e(t) := \pi - \alpha(t)$ and show that $e(t) \to 0$ exponentially.

It is straightforward to rewrite all the functions of α in terms of functions of the new variable e, e.g.,

$$\tilde{N}(e(t)) := 2R\sin(e(t)) + 4R\sqrt{2e(t) - \sin(2e(t))}. \quad (5)$$

Moreover, we find that the dynamics of e is given by

$$\begin{aligned} \dot{e} = -\dot{\alpha} = -\frac{1}{2}\omega &= -\frac{1}{2R}\tilde{N}(e) \\ &= -(\sin(e) + 2\sqrt{2e - \sin(2e)}) =: -\phi(e). \end{aligned}$$

An analysis of the function

$$\phi : [0, \pi] \to \mathbb{R}, \quad e \mapsto \sin(e) + 2\sqrt{2e - \sin(2e)} \quad (6)$$

shows that ϕ is a *sector nonlinearity* with

$$e \leq \phi(e) \leq 3e.$$

The lower bound immediately yields

$$e(t) \leq e^{-t}e(0).$$

Next we show the claimed exponential decay for

$$N(x(t), y(t), z(t)) = \tilde{N}(e(t)).$$

Starting with the relationship

$$\tilde{N}(e(t)) = 2R\phi(e(t)),$$

which can be readily inferred from comparing the two equations (5) and (6), we first use $\phi(e) \leq 3e$ to obtain

$$\tilde{N}(e(t)) \leq (2R) \times 3\, e^{-t}e(0).$$

Applying $e \leq \phi(e)$, we have

$$\tilde{N}(e(t)) \leq (2R) \times 3\, e^{-t}\phi(e(0)).$$

Since $2R\phi(e(0)) = \tilde{N}(e(0))$, we arrive at the claim

$$\tilde{N}(e(t)) \leq 3\, e^{-t}\tilde{N}(e(0)). \quad \square$$

Remark 1 *We emphasize that the specific definition of the gain $N(x, y, z)$ is what facilitates that the resulting dynamics for e is independent of R, which is critical for obtaining a* uniform *bound of the form*

$$N(x(t), y(t), z(t)) \leq 3\, e^{-t}\, N(x(0), y(0), z(0)).$$

In more detail, the specific choice of the orders in $\sqrt{x^2 + y^2}$ and $\sqrt{|z|}$ is what facilitates that both terms in (5) *contain a factor of R that is cancelled when considering the dynamics $\dot{e} = -\frac{1}{2R}\tilde{N}(e) = -\phi(e)$.*

At the same time, this particular choice is also ultimately the cause of the failure of N to qualify as a proper norm due to a lack of absolute homogeneity.

Figure 5 shows the evolution of the nonholonomic integrator under the proposed state feedback law for the initial state $(x(0), y(0), z(0)) = (0, 0, 1)$. In this specific instance, we can also see that $N(x, y, z)$ is actually increasing in the beginning before it starts decreasing. This provides further context regarding the need for a more specialized treatment instead of a standard Lyapunov-based argument in the foregoing proof.

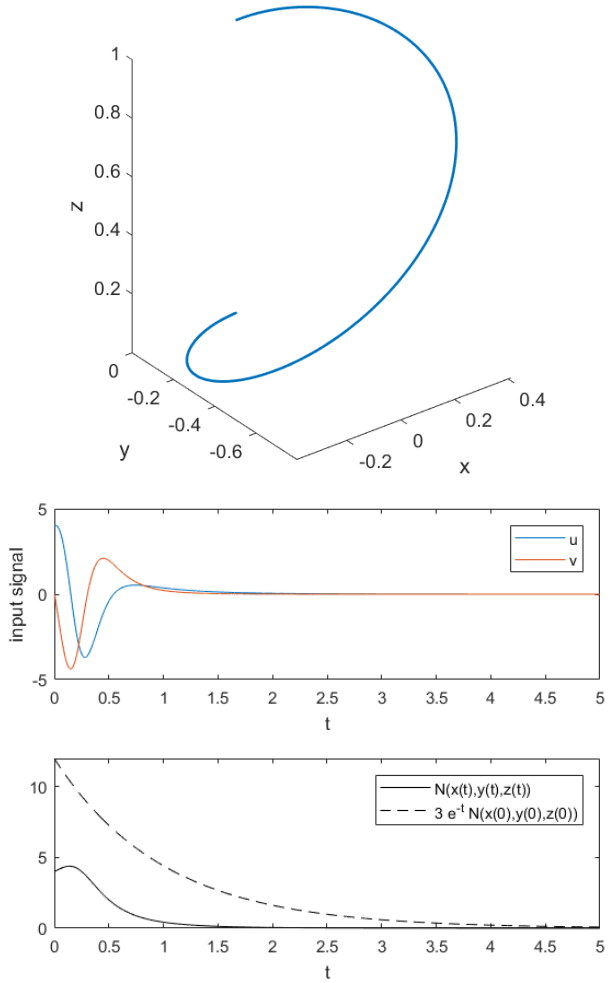

Fig. 5. Top: The closed-loop state trajectory. Middle: The resulting input signals u and v. Bottom: The evolution of $N(x(t), y(t), z(t))$ (solid) along with the established upper bound given by $3\,e^{-t}N(x(0), y(0), z(0))$ (dashed).

4 Summary and discussion

We have revisited the classical example of the nonholonomic integrator and examined it in an elementary geometric light. This rather naturally led to the elementary construction of a geometric solution the problem of stabilizing the nonholonomic integrator with time-invariant state feedback. We built on the geometric insights and established a less intuitive uniform exponential convergence result by employing systems theoretic arguments that go beyond the purely geometrical considerations.

From a broader perspective, the results of this paper highlight interesting points in the control of certain nonholonomic control systems, such as the possibility of even the most natural seeming feedback law to fundamentally refuse a description in terms of a closed expression.

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