A result by Nazarov, Peres, and Volberg

Krystal Taylor

Given a set E in the plane, one may consider the probability that "Buffon's needle," a long line segment dropped at random, intersects the set. In the case that E is a neighborhood of the four-corner Cantor set, a theorem of Besicovitch implies that this probability tends to zero. We discuss a result due to Nazarov, Peres, and Volberg, which gives an explicit upper bound for the rate of decay in terms of the Favard length.

What is the four-corner Cantor set?

Replace the unit square by a subset consisting of the 4 corner squares of side length $\frac{1}{4}$. Repeat in a self-similar manner. Then,

$$\mathcal{K} = \cap \mathcal{K}_n.$$

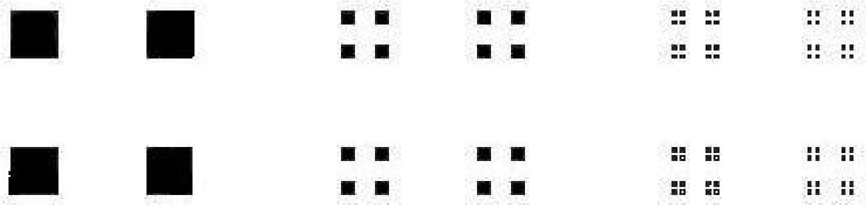

Figure: $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$. Stages 1, 2, and 3 in the construction of square Cantor set. $\mathcal{K}_n$ is made up of 4^n squares of side length $\frac{1}{4^n}$.

We are interested in the probability that “Buffon’s needle”- a long, straight needle with independent and uniform distribution on its orientation and distance from the origin will intersect $\mathcal{K}$.

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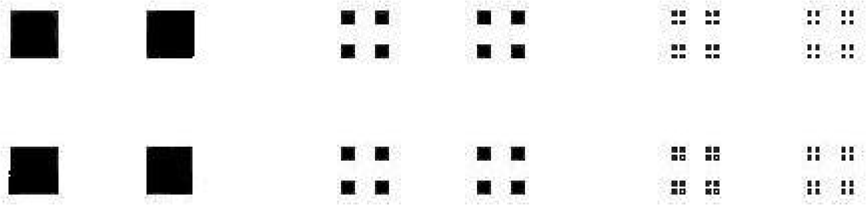

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Hausdorff measure, Hausdorff dimension

$$\mathcal{H}^s(E) = \lim_{\delta \to \infty} \inf\{\sum_{i=1}^{\infty} |U_i|^s\},$$

where $E \subset \cup_{i=1}^{\infty} U_i$, $|U_i| \leq \delta$.

$\alpha > s \to \mathcal{H}^\alpha(E) = 0$
$\alpha < s \to \mathcal{H}^\alpha(E) = \infty$

Another notion of length

The **Favard length** of a set $E \subset \mathbb{R}^2$ is defined by

$$Fav(E) = \frac{1}{\pi} \int_0^{\pi} |ProjR_\theta E| \, d\theta$$

Examples:

- $\theta = 0$, then $|ProjR_\theta(K_n)| = \frac{1}{2^n}$
- $\theta = \arctan 1/2$, then $|ProjR_\theta(K_n)| \sim 1$

Conclusion: Sometimes lots of stacking occurs, sometimes no stacking occurs.

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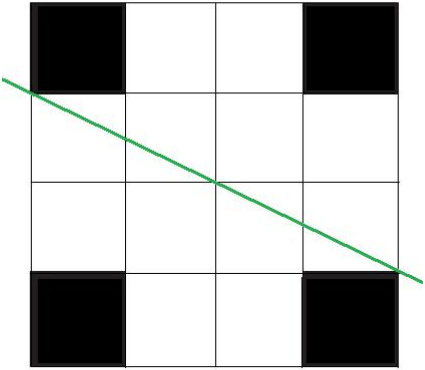
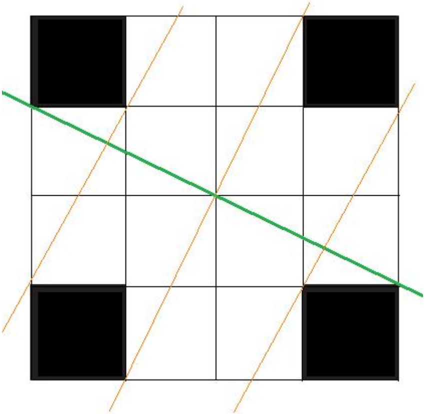
For some angles, there is no stacking

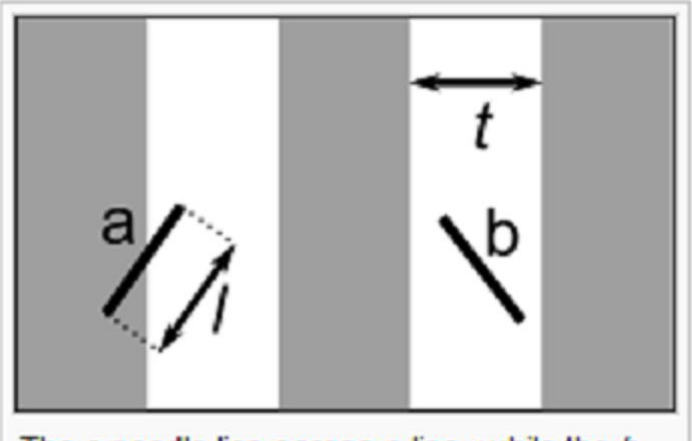




Where did “Buffon’s needle probability” come from?

In the 18th century, Count Buffon posed the following geometric probability problem:

Consider a striped floor with length, t, between the strips, and drop a needle of length l onto the floor. What is the probability that needle crosses a strip?



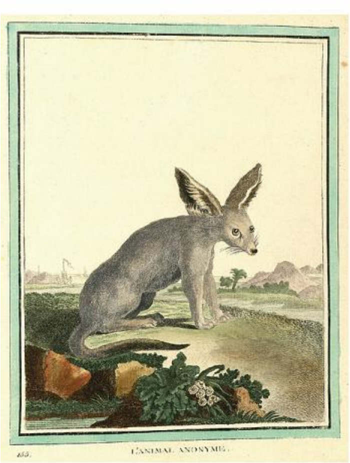


Figure 1: One of Count Buffon's beasts.

Figure: One of Count Buffon's Beasts. Possibly made up by Matt Bond.

The problem can be made more general by replacing the striped floor with a bounded set $E \subset \mathbb{R}^2$, and replacing the needle by an infinite line with independent and uniform distribution on its position and orientation.

In recent years, Buffon's needle probability of Cantor sets has gained interest. The Four-Corner Cantor set appeared as a counter-example to well known conjecture. The study of the Buffon needle probability in the Four-Corner Cantor set has connections to the construction of Kakeya sets.

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Painleve problem: find geometric characterization of removable sets

$\gamma(E) = \{\sup_{z\to\infty} |zf(z)|\}$ where sup is taken over f satisfying:
$f \in Hol(\mathbb{C}\backslash E)$, $\|f\|_\infty < \infty$, $f(\infty) = 0$.

Conjecture: $\mathcal{H}^1(E) = 0 \iff \gamma(E) = 0$

The forward direction is true!

Vituskin and Garnett show that the converse is FALSE!
Counter Example: quarter-Cantor set

Quantitative estimate: Tolsa proved that $\gamma(\partial(K_n))$ decays like $\frac{1}{\sqrt{n}}$.

Notice: $\mathcal{H}^1(\text{square}) = \infty$, $\mathcal{H}^1(\partial(\text{square})) \sim 1$

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Reformulated conjecture with new notion of length

A theorem of Besicovitch shows that the projection of the four-corner Cantor set, $\mathcal{K}$, to almost every line through the origin has zero length. i.e.

$$Fav(\mathcal{K}) = 0.$$

Conjecture (Vitushkin): $Fav(E) = 0 \iff \gamma(E) = 0$

Since $Fav(\mathcal{K}) = 0$, the dominated convergence theorem implies that $\lim n \to \infty Fav(\mathcal{K}_n) = 0$.

What is the exact rate of decay?

We state several results which give either upper or lower bounds on the rate of decay of the Favard length of the $n - th$ iteration in the construction of $\mathcal{K}$ with our emphasis being on the result of Nazarov, Peres, and Volberg.

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How does Favard length fit into our probability question?

The Favard length of a set E, sometimes called the **Buffon needle Probability**, has a probabilistic interpretation. Up to a constant factor, it is the probability that Buffon's needle will intersect E.

To see this, notice that a line, l, intersects E if and only if l intersects the orthogonal projection of E onto any line perpendicular to l.

Vitushkin's conjecture does not always hold

Conjecture (Vitushkin): $Fav(E) = 0 \iff \gamma(E) = 0$
Vituskin's conjecture is sometimes true.

$\mathcal{H}^1(E) = 0 \rightarrow$ True
$\mathcal{H}^1(E) \in (0, \infty) \rightarrow$ True (NTV, David)
$\mathcal{H}^1(E) = \infty \rightarrow$ counter example to the forward direction (Jones, Murai), converse direction is OPEN.

Statements of Background Results

Theorem

(Mattila, 1990)

$$Fav(E_n) \gtrsim \frac{1}{n}.$$

Theorem

(Peres and Solomyak, 2002)

$$Fav(E_n) \lesssim e^{-c \log^* n},$$

where $\log^* n$ *denotes the number of times that one must iterate the* $\log$ *function to get from* n *to* 1.

Theorem

(Bateman and Volberg, 2008)

$$Fav(\mathcal{K}_n) \gtrsim \frac{\log n}{n}.$$

Nazarov, Peres, and Volberg 2008

Theorem

For every $\delta > 0$, there exists $C > 0$ such that

$$Fav(\mathcal{K}_n) \leq Cn^{\delta - 1/6}, \qquad \text{for all } n \in \mathbb{N}. \tag{1}$$

Laba and Zhai, 2009

Theorem

Assume that for some direction θ_0, $|Proj(R_{\theta_0}(E_\infty))| > 0$. *Then*

$$Fav(E_n) \leq Cn^{-1/p},$$

for some $p \in (6, \infty)$ *which depends on the choice of* K, A, *and* B. *The explicit range of* p *is described in, "The favard length of product cantor sets".*

Generalized Cantor Set

$A, B \subset \{0, 1, \cdots, K-1\}$

$|A||B| = K$, $|A| > 1$, $|B| > 1$

Divide the planar unit square into K^2 congruent squares, and keep those squares with bottom left vertex in $(\frac{A}{K} \times \frac{B}{K})$.
Iterate the process.

Skeleton of proof:

- Fix n and θ. Define a counting function, $f_{n,\theta}$.
- State two lemmas

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Where is the work done to generalize the NPV argument?

The counting function

- Fix θ and n.
- Recall $R_\theta(\mathcal{K}_n)$ consists of 4^n squares, and $ProjR_\theta(\mathcal{K}_n)$ is a union of 4^n intervals, possibly with much overlap.
- Use trigonometry to find center and length of intervals.
- Define f_n as sum of characteristic functions of intervals.
- $f_n(x) =$ the number of squares of $R_\theta(\mathcal{K}_n)$ which project onto x.
- Re-write f_n using $\chi_{[c-l,c+l]} = \chi_{[-l,l]} * \delta_c$.

$$f_n = \nu^n * \frac{4^n}{\rho}\chi_{[-\frac{\rho}{2}4^{-n},\frac{\rho}{2}4^{-n}]}$$

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Lemma (Fourier-analytic part)

Let K large, S large, and

$$E = \left\{ \theta : \max_{1 \le n \le (KS)^q} \int_{\mathbb{R}} f_n^2 \le K \right\}.$$

Then $|E| \lesssim \frac{1}{S}$ whenever $q > 4$.

Note: If θ satisfies $\int_{\mathbb{R}} f_n^2 \le K$, then $|ProjR_\theta(\mathcal{K}_n)| \ge \frac{1}{K}$. We will need opposite containment.

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First reduction

Lemma (Combinatorial/ Self-similarity part)

Let K, S large. Let $N \geq K^p S^q$ with $p > 6$ and $q > 4$. Then,

$$\left|\left\{\theta \in [0, \frac{\pi}{4}] : |ProjR_\theta(\mathcal{K}_N)| \geq \frac{1}{K}\right\}\right| \lesssim \frac{1}{S}.$$

Now,

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} |ProjR_\theta(\mathcal{K}_n)| d\theta &= \int_0^{\infty} \left|\{\theta \in (0, \frac{\pi}{4}) : |ProjR_\theta(\mathcal{K}_n)| \geq t\}\right| dt \\
&= \int_0^{\sqrt{2}} \left|\{\theta \in (0, \frac{\pi}{4}) : |ProjR_\theta(\mathcal{K}_n)| \geq t\}\right| dt \\
&\lesssim \int_0^{CN^{-1/p}} 1 dt + \int_{CN^{-1/p}}^{\sqrt{2}} N^{-1/q} t^{-p/q} dt \\
&\lesssim N^{-1/p}
\end{aligned}$$

Sketch proof of Fourier-analytic lemma

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Begin by noticing that if $t \in E$, then

$$K \geq \int_{\mathbb{R}} |f_N(x)|^2 dx = \int_{\mathbb{R}} |\widehat{f_N}(y)|^2 dy \gtrsim \int_1^{4^{N/2}} |\widehat{\nu}^N(y)|^2 dy.$$

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Lemma

For $1 \leq m, n \leq N/10$, *define*

$$I = \int_{4^{n-m}}^{4^n} \left| \widehat{\nu}^N(y) \right|^2 dy.$$

For each $1 \leq n \leq N/10$ *and a set* $E^* \subset E$ *with* $|E^*| \geq |E|/2$ *such that for all* $t \in E^*$ *we have*

$$I \leq \frac{8Km}{N}.$$

The idea now is to get a lower bound for I and to show that we meet a contradiction of the lemma above unless $|E| \lesssim \frac{1}{S}$.

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To get a lower bound on I, write

$$
\begin{aligned}
I :=& \int_{4^{n-m}}^{4^n} \left|\widehat{\nu}^N(y)\right|^2 dy \\
\sim& 4^n \int_{4^{-m}}^{1} \left|\prod_{k=0}^{n} \frac{\cos 4^k y + \cos 4^k ty}{2}\right|^2 dy \\
=& 4^n \int_{4^{-m}}^{1} \left|\prod_{k=0}^{m} \frac{\cos 4^k y + \cos 4^k ty}{2}\right|^2 \left|\prod_{k=m+1}^{n} \frac{\cos 4^k y + \cos 4^k ty}{2}\right|^2 dy \\
=& 4^n \int_{4^{-m}}^{1} |P_1(y)|^2 \, |P_2(y)|^2 \, dy \\
\gtrsim& 4^n (4^{-2m} S^{-2} 4^{-2m})^2 \int_{[4^{-m},1]\setminus SSV} |P_2(y)|^2 \, dy
\end{aligned}
$$

- Let SSV denote the set of small values of $|P_1|$.
  ($|P_1| > 4^{-2m}(S^{-2}4^{-2m})$ on $[4^{-m}, 1]\backslash SSV$.)
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  (proof of this fact is generic)
- Lemma B: $\frac{1}{|E^*|}\int_{E^*}(\int_{SSV} |P_2(y)|^2 dy) dt << 4^{m-n}$
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The lemma is proved by contradiction and the idea is to get an appropriate lower bound on

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Obstacle: The integrand has mass concentrated near zero, but we are integrating away from 0.

- $$\int_{4^{-m}}^{1} \Big| \prod_{k=m+1}^{n} \frac{\cos 4^k y + \cos 4^k ty}{2} \Big|^2 dy \gtrsim 4^{m-n}$$
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To prove

- $$\int_{4^{-m}}^{1} \Big| \prod_{k=m+1}^{n} \frac{\cos 4^k y + \cos 4^k ty}{2} \Big|^2 dy \gtrsim 4^{m-n},$$

begin by writing

$$\frac{\cos 4^k y + \cos 4^k ty}{2} = 4^{m-n} \sum_{j=1}^{4^{n-m}} e^{i\lambda_j y},$$

where $\lambda_j = \sum_{k=m+1}^{n} 4^k \xi_k$ for $\xi_k \in \{\pm 1, \pm t\}$. Then the estimate reduces to showing that

$$\int_{[-1,1]\setminus[-4^{-m},4^{-m}]} \Big| \sum_{j=1}^{4^{n-m}} e^{i\lambda_j y} \Big|^2 dy \gtrsim 4^{n-m}.$$

To give the flavor of the argument, we look at a simplified problem. That is to show that

$$\int_{[-1,1]} |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 dy \gtrsim 4^{n-m}.$$

We will show that

$$\int_{[-1,1]} |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 h(y) dy \gtrsim 4^{n-m}.$$

Write

$$
\begin{aligned}
&\int_{[-1,1]} |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 h(y)dy \\
=&\sum_{j=1}^{4^{n-m}}\sum_{k=1}^{4^{n-m}} \int_{[-1,1]} e^{-i(\lambda_j-\lambda_k)y} h(y)dy \\
=&\sum_{j=1}^{4^{n-m}}\sum_{k=1}^{4^{n-m}} \widehat{h}((\lambda_j-\lambda_k)) \\
=&4^{n-m}\widehat{h}(0)) + \sum_{j\neq k}^{4^{n-m}} \widehat{h}((\lambda_j-\lambda_k)) \\
\gtrsim&4^{n-m}
\end{aligned}
$$

To show that

$$\int_{[-1,1]\setminus[-4^{-m},4^{-m}]} |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 dy \gtrsim 4^{n-m},$$

we introduce a slightly more complicated function. Properties of g:
-g is supported on $[-1,1]\setminus[-4^{-m},4^{-m}]$
-$g \leq 1$
-$\int_{\mathbb{R}} g \geq \frac{1}{2}$
-g is even
-$\widehat{g} \geq -c\frac{L}{\lambda^2+L^2}$.

- Write down zeros of

$$\prod_{k=0}^{m+1} \frac{\cos 4^k y + \cos 4^k t y}{2}$$

Sketch proof of Self-Similarity lemma

Fix θ, K, N_0.
Define

$$\nu_\theta = \left| \left\{ x : \max_{1 \le n \le N_0} f_n(x) \ge K \right\} \right| .$$

"ν_θ is the *size* of the set of x with lots of stacking".

- $\nu_\theta \gtrsim \frac{1}{K^3} \to |ProjR_\theta(\mathcal{K}_N)| \le \frac{1}{K}$ for $N > K^p S^q$.
  "Lots of stacking, then size of projection smaller"
- $\nu_\theta \lesssim \frac{1}{K^3} \to \max_{1 \le n \le N_0} \int f_n^2 \le 2K$, i.e. $\theta \in E$.

Conclusion: $\{\theta : |ProjR_\theta(\mathcal{K}_N)| \ge \frac{1}{K}\} \subset E$ when N sufficiently large.

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In order to show that $\nu_\theta \lesssim \frac{1}{K^3} \to \max_{1 \leq n \leq N} \int f_n^2 \leq 2K$, we will need:

Lemma

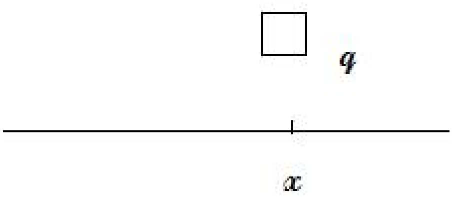
Let K, M, N large. Let $F^*(x) = \max_{1 \leq n \leq N} f_n(x)$. Then,

$$|\{x : F^*(x) \geq 4KM\}| \leq 108K |\{x : F^*(x) \geq K\}| |\{x : F^*(x) \geq M\}|.$$

Preliminary observations:

- $f_n(x) \leq 2 f_{n-1}(x)$
- $|\{x : F^*(x) \geq M\}| = \frac{1}{|Q|} |\{x : F_Q^*(x) > M\}|$

Observation 1: $f_n \leq 2f_{n-1}$



Conclusion: If n is the first generation for which $f_n(x) \geq 2K$, then

$$2K \leq f_n(x) < 4K.$$

Observation 2: $|\{x : F^*(x) \geq M\}| = \frac{1}{|Q|}|\{x : F^*_Q(x) > M\}|$

Let Q be a square in $\mathcal{K}_i$.

Recall $F^*(x)$ as the maximum over $1 \leq n \leq N_0$ of the number of squares of generation n which project onto x.

Define $F^*_Q(x)$ as the maximum over $(i+1) \leq n \leq N_0 + (i+1)$ of the number of squares of generation n which both project onto x and are contained in Q.

Then,

$$|\{x : F^*(x) \geq M\}| = \frac{1}{|Q|}|\{x : F^*_Q(x) > M\}|.$$

Proof.

Consider $\{x : F^*(x) \geq 2K\}$.
Let $n(x)$ minimal generation for which $f_{n(x)}(x) \geq 2K$.
Then, $2K \leq f_{n(x)}(x) < 4K$.
Mark these squares.
Take maximally marked squares: $Q_1, \cdots, Q_J$.
Consider $\{x : F^*(x) \geq 4KM\}$.
Let $x \in \{x : F^*(x) \geq 4KM\}$.
Pigeon hole: $\exists$ a square of generation $n(x)$ with M descendents, all of same generation, each of which projects onto x.
$\exists$ Q_i, a M.M.S., with M descendents, all of same generation, each of which projects onto x.
Conclude: $\{x : F^*(x) \geq 4KM \cap ProjQ_i\} \subset \{x : F^*_{Q_i}(x) > M\}$.

□