1 The Power Law For The Buffon Needle Probability Of The Four-Corner Cantor Set

after F. Nazarov, Y. Peres, and A. Volberg [5]
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Abstract

Given a set E in the plane, one may consider the probability that "Buffon's needle," a long line segment dropped at random, intersects the set. In the case that E is a neighborhood of the four-corner Cantor set, a theorem of Besicovitch implies that this probability tends to zero. We discuss a result due to Nazarov, Peres, and Volberg, which gives an explicit upper bound for the rate of this decay in terms of the Favard length [5]. In a sequential paper, Laba and Zhai improve on the techniques in [5] to include a larger class of Cantor Sets [2]. We state these and several other result, and look at some of the key steps in the proof.

2 Introduction

2.1 The context: analytic capacity, hausdorff measure, and favard length

The Painleve problem asks one to find a geometric characterization of certain removable sets for bounded analytic functions. In 1947, Ahlfors introduced the notion of the analytic capacity of a compact set $E \subset \mathbb{C}$. He showed that the analytic capacity of E, denoted $\gamma(E)$, is zero if and only if E is removable for bounded analytic functions [6]. There have been steps forward in an attempt to understand the situation geometrically, and it was conjectured that the one dimensional Hausdorff measure of E, denoted by $\mathcal{H}^1(E)$, equals zero if and only if $\gamma(E) = 0$. While an argument using complex analysis shows that the forward direction of this conjecture is true, there is a complicated counter example due to Vitushkin for the converse. Later, Garnett found a counter example which is simpler to describe. He observed that when $E = \mathcal{K}$, the four-corner Cantor set, then $\mathcal{H}^1(\mathcal{K}) > 0$ while $\gamma(\mathcal{K}) = 0$ [6] [4].

In light of these counter examples, the conjecture was then re-stated with a new notion of length. Vitushkin conjectured that the Favard length of a set E is zero if and only if $\gamma(E) = 0$. Although this conjecture turns out to not always be true, the four-corner Cantor set, $\mathcal{K}$, has zero Favard length. In this summary, we state several results which give either upper or lower bounds on the rate of decay of the Favard length of the $n-th$ iteration in the construction of $\mathcal{K}$ with our emphasis being on the result of Nazarov, Peres, and Volberg.

2.2 The set up: definitions and notation

To construct the four-corner Cantor set, $\mathcal{K}$, begin by replacing the unit square with four sub-squares of side length $\frac{1}{4}$ located at the corners of the unit square. Then, repeat this process indefinitely within each sub-square in a self-similar manner with a scaling factor of $\frac{1}{4}$. Let $\mathcal{K}_n$ denote the set which comes from the $n-th$ iteration of this process; $\mathcal{K}_n$ is a union of 4^n squares of side length $\frac{1}{4^n}$. Then, $\mathcal{K} = \cap \mathcal{K}_n$.

To study the probability that the "Buffon's needle" (an infinite line with direction chosen uniformly at random and then located in a uniformly chosen position in that direction, at a distance at most, say, $\sqrt{2}$ from the origin) intersects a neighborhood of the 4-corner Cantor set, mainly $\mathcal{K}_n$ for some fixed n, we will use the notion of Favard length.

The ***Favard length*** of a set $E \subset \mathbb{R}^2$ is defined by

$$Fav(E) = \frac{1}{\pi} \int_0^{\pi} |ProjR_\theta E| d\theta, \tag{1}$$

where Proj denotes the orthogonal projection from $\mathbb{R}^2$ to the horizontal axis, and where R_θ is the counterclockwise rotation by an angle θ.

The Favard length of a set E is sometimes called the ***Buffon needle Probability*** of the set because, up to a constant factor, it is the probability that Buffon's needle will intersect E. To see this, notice that a line l intersects E if and only if l intersects the orthogonal projection of E onto any line perpendicular to l.

A theorem of Besicovitch implies that the projection of $\mathcal{K}$ to almost every line through the origin has zero length. This means that $Fav(\mathcal{K}) = 0$. A recent result of Nazaroz, Peres, and Volberg reveals that $Fav(\mathcal{K}_n) = O(n^{-\frac{1}{6}})$. [5].

3 Results

3.1 Statements of background results

Let $A_n \lesssim B_n$ mean that there exists a constant C, which is independent of n, so that $A_n \leq CB_n$. If $A_n \lesssim B_n$ and $B_n \lesssim A_n$, then we write $A_n \sim B_n$.

In 1990, Mattila showed that $Fav(E_n) \gtrsim \frac{1}{n}$. In 2002, Peres and Solomyak proved that $Fav(E_n) \lesssim e^{-c\log^* n}$, where $\log^* n$ denotes the number of times that one must iterate the log function to get from n to 1. Next, in 2008, Bateman and Volberg showed that $Fav(\mathcal{K}_n) \gtrsim \frac{\log n}{n}$.

3.2 Statements of the main result and a generalization

The main result of this summary follows.

Theorem 1. *(Nazarov, Peres, and Volberg 2008) For every $\delta > 0$, there exists $C > 0$ such that*

$$Fav(\mathcal{K}_n) \leq Cn^{\delta - 1/6}, \quad \textit{for all } n \in \mathbb{N}. \tag{2}$$

A generalization of the previous theorem follows.

Theorem 2. *(Laba and Zhai, 2009) Let E_∞ be a generalized Cantor set. Assume that for some direction θ_0, $|Proj(R_{\theta_0}(E_\infty))| > 0$. Then there exists a $p \in (6, \infty)$, which depends on the choice of K, A, and B, so that*

$$Fav(E_n) \leq Cn^{-1/p}.$$

The explicit range of p is described in [2].

3.3 Sketch of the proof of 1

The proof of (2) is is divided into a two sections; the first section deals with a harmonic-analytic estimate, and the second section takes advantage of self-similarity in a combinatorial estimate. Both sections rely on a function which is, up to some minor re-scaling, the sum of the characteristic functions of the projections of $\mathcal{K}_n$ for a fixed rotation and a fixed value of n.

3.3.1 The counting function

Finding the right notation to write down a function which is the sum of the characteristic functions of the projections of $R_\theta \mathcal{K}_n$ can be a bit tedious. This section shows a simple way to write the desired function, with some minor re-scaling, in terms of convolutions.

To begin, we re-center $\mathcal{K}_n$ by replacing it with the set $\mathcal{K}_n - (\frac{1}{2}, \frac{1}{2})$. Due to symmetries, it is enough to average over $\theta \in (0, \frac{\pi}{4})$ in defintion (1) of $Fav(\mathcal{K}_n)$. Now, the projection of $R_\theta(\mathcal{K}_n - (\frac{1}{2}, \frac{1}{2}))$ to the horizontal axis is the union of 4^n intervals of length $4^{-n}(\cos\theta + \sin\theta)$ centered at the points $\sum_{k=0}^{n-1} 4^{-k}\xi_k$, where $\xi_k \in \{\pm\frac{3\sqrt{2}}{8}\cos\theta, \pm 3\frac{\sqrt{2}}{8}\sin\theta\}$. For the sake of notation, we notice that

$$\begin{aligned}
&\Big|Proj(R_\theta(\mathcal{K}_n - (\frac{1}{2}, \frac{1}{2}))\Big| \\
&= \Big| \cup \Big[\sum_{k=0}^{n-1} 4^{-k}\xi_k - \frac{4^{-n}(\cos\theta + \sin\theta)}{2}, \sum_{k=0}^{n-1} 4^{-k}\xi_k + \frac{4^{-n}(\cos\theta + \sin\theta)}{2} \Big] \Big| \\
&= \cos(\frac{\pi}{4} - \theta)\frac{3\sqrt{2}}{8} \Big| \cup \Big[\sum_{k=0}^{n-1} 4^{-k}\eta_k - \frac{4^{-n}\rho}{2}, \sum_{k=0}^{n-1} 4^{-k}\eta_k + \frac{4^{-n}\rho}{2} \Big] \Big|,
\end{aligned}$$

where $\eta_k \in \{\pm 1, \pm\tan(\frac{\pi}{4} - \theta)\}$, $\rho = \frac{8}{3\sqrt{2}}\left(\frac{\cos\theta + \sin\theta}{\cos(\frac{\pi}{4} - \theta)}\right)$, and ξ_k is defined above. This shows that the length of the projection is comparable to the union of 4^n intervals of length $4^{-n}\rho$ centered at the points $\sum_{k=0}^{n-1} 4^{-k}\eta_k$. We are now ready to define f_n. Let $t = \tan(\frac{\pi}{4} - \theta)$. For $\theta \in [0, \frac{\pi}{4})$, define

$$f_n = \sum_{\eta \in \{\pm 1, \pm t\}} \chi_{\left[\sum_{k=0}^{n-1} 4^{-k}\eta_k - \frac{4^{-n}\rho}{2}, \sum_{k=0}^{n-1} 4^{-k}\eta_k + \frac{4^{-n}\rho}{2}\right]}.$$

Since $\chi_{[c-r, c+r]} = \delta_c * \chi_{[-r,r]}$,

$$f_n = \nu^{(n)} * \frac{4^n}{\rho}\chi_{[-\frac{\rho}{2}4^{-n}, \frac{\rho}{2}4^{-n}]},$$

where $\nu^{(n)} = *_{k=0}^{n-1}\nu_k$ and $\nu_k = \frac{1}{4}[\delta_{-4^{-k}} + \delta_{-4^{-k}t} + \delta_{4^{-k}t} + \delta_{-4^{-k}}]$.

It will be useful to observe that $\int_{\mathbb{R}} |\widehat{f_n}(y)| \gtrsim \int_1^{4^{n/2}} |\widehat{\nu^{(n)}}(y)|$ for n sufficiently large, because $\psi = \frac{4^n}{p}\chi_{[-\frac{\rho}{2}4^{-n}, \frac{\rho}{2}4^{-n}]}$ satisfies $\widehat{\psi}(y) \gtrsim 1$ for all $|y| < 4^{n/2}$.

3.3.2 Fourier-analytic part

Let K and S be large positive numbers. Then for $q > 4$, the set

$$E = \{t \in [0,1] : \max_{1 \leq n \leq (KS)^q} \int f_n^2 \leq K\} \tag{3}$$

satisfies $|E| \leq \frac{1}{S}$.

One of the key estimate in the proof of (3) which sets this argument apart from that of the general case is that one can identify the zero set of $\prod_{k=0}^{m} \frac{\cos 4^k y + \cos 4^k ty}{2}$, where this product arises upon re-writing $\widehat{\nu}$ in terms of cosines.

One of the main estimates in the proof of (3), is to show that

$$\int_{4^{-m}}^{1} |\prod_{k=m+1}^{n} \frac{\cos 4^k y + \cos 4^k ty}{2}|^2 dy \gtrsim 4^{m-n} \tag{4}$$

where $m \leq n < \frac{1}{2}(KS)^q$ is choosen so that 4^m is a large multiple of K.

Showing (4) reduces to showing that

$$\int_{4^{-m}}^{1} |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 dy \gtrsim 4^{n-m}. \tag{5}$$

Next, the idea is to introduce a function g on $\mathbb{R}$ with the following properties:
-g is even
-$\int_{\mathbb{R}} g \geq \frac{1}{2}$
-g is supported on $[-1,1] \backslash [-4^{-m}, 4^{-m}]$
-$g \leq 1$
-$\widehat{g} \geq -c\frac{L}{\lambda^2 + L^2}$ with some constant $c > 0$ and $L = 4^m$. This function is given explicitly in [5].

Now

$$\int_{[-1,1]\backslash[-4^{-m},4^{-m}]} |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 dy \geq \int_{\mathbb{R}} g(y) |\sum_{j=1}^{4^{n-m}} e^{i\lambda_j y}|^2 dy$$

To bound the above quantity above, we will take advantage of the fact that for $t \in E$, $\int_{\mathbb{R}} f_{n-m}^2 \leq K$ and, with a change of variable, this is equivalent to

$$\int_{\mathbb{R}} (\sum_j \chi_{[\lambda_j - \frac{\varrho}{2}4^m, \lambda_j + \frac{\varrho}{2}4^m]})^2 \leq K4^n. \tag{6}$$

Properties of the Poisson kernel play a role.

3.3.3 Finishing the proof

To finish the proof the theorem (1), it suffices to show for $t > 0$ that

$$|\{\theta \in (0, \frac{\pi}{4}) : |ProjR_\theta \mathcal{K}_N| \geq t\}| \lesssim (N^{-1}t^{-p})^{1/q}, \tag{7}$$

where $p > 6$ and $q > 4$.

Indeed, (7) implies that

$$\begin{aligned}\int_0^{\frac{\pi}{4}} |ProjR_\theta \mathcal{K}_n| d\theta &= \int_0^{\infty} |\{\theta \in (0, \frac{\pi}{4}) : |ProjR_\theta \mathcal{K}_n| \geq t\}| dt \\ &= \int_0^{cN^{-1/p}} 1 dt + \int_{cN^{-1/p}}^{\sqrt{2}} (N^{-1}t^{-p})^{1/q} dt \\ &\lesssim N^{-1/p}.\end{aligned}$$

We prove that if K and S large enough and $N \geq K^p S^q$, with $p > 6$ and $q > 4$, then

$$|\{\theta \in (0, \frac{\pi}{4}) : |ProjR_\theta \mathcal{K}_N| \geq \frac{C}{K}\}| \lesssim \frac{1}{S}. \tag{8}$$

The idea behind the proof of (8), is to first fix θ, and then consider the set of x where much stacking occurs for that choice of θ. When this set is small, we show that θ is in a set of small measure. When this set is larger, we show that, beyond a large generation, the size of the projection, $|ProjR_\theta \mathcal{K}_N|$ is small. We combine these observations with the Fourier-analytic part to conclude that the set on the left-hand-side of (8) is small. It is within this proof that the condition that $p > 6$ arises.

In more detail, fix θ, K, and N. Set $F^*(x) = \max_{1 \leq n \leq N} f_n(x)$, and let $\nu = |\{F^* \geq K\}|$. When θ is such that $\nu \lesssim \frac{1}{K^3}$, it can be shown that $\theta \in E$; here we will need $N \geq (KS)^q$ where $q > 4$. When θ is such that $\nu \gtrsim \frac{1}{K^3}$, it can be shown that $|ProjR_\theta \mathcal{K}_N| \lesssim \frac{1}{K}$; here we will need $N \geq K^p S^q$ where $p > 6$ and $q > 4$. We conclude that

$$\{\theta \in (0, \frac{\pi}{4}) : |ProjR_\theta \mathcal{K}_N| \geq \frac{C}{K}\} \subset E.$$

Since the Fourier-analytic part of the argument shows that $|E| \leq \frac{1}{S}$, this concludes (8).

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