Math 5201 Analysis Fall 2015

Homework Assignment #10 November 12 Solutions

1. Rudin, Chapter 6 #1

   Solution It suffices to take a partition with four points: $t_0 = a$, $t_1 < x_0 < t_2$ and $t_3 = b$. On $[t_0, t_1]$ and on $[t_2, t_3]$ we have $M_i = m_i = 0$, so

   $$U(P, f, \alpha) - L(P, f, \alpha) = \alpha(t_2) - \alpha(t_1)\,.$$

   Now, since α is continuous at x_0, then given any $\varepsilon > 0$, there is a $\delta > 0$ such that when $x_0 - \delta < t_1 < x_0$ and $x_0 < t_2 < x_0 + \delta$ then $\alpha(t_2) - \alpha(t_1) < \varepsilon$.

   Hence, choosing t_1 and t_2 to satisfy those bounds, we have $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for that partition, and by Theorem 6.6, $f \in \mathcal{R}(\alpha)$.

   Also, since $m_i = 0$ for all i, we have $L(P, f, \alpha) = 0$, and so $\int f\, d\alpha = 0$.

2. Rudin, Chapter 6 #2

   Solution We prove this by contradiction.

   Suppose that $f(x_0) = a > 0$ at some point, x_0. Then, taking $\varepsilon = a/2$, by continuity of f there is a $\delta > 0$ such that $f(x) > a/2$ on $(x_0 - \delta, x_0 + \delta)$. Take a partition $P = \{t_0, t_1, t_2, t_3\} = \{a, x_0 - \delta, x_0 + \delta, b\}$. Since $m_i \geq 0$ for all i (as $f \geq 0$), we have

   $$L(P, f) \geq a\delta > 0\,.$$

   Since we have constructed a partition with $L(P, f) > 0$, then clearly the upper bound of $L(P, f) > 0$, which contradicts the statement that $\int f = 0$.

   Hence, $f \equiv 0$.

   This contrasts with Problem 1 in two ways. First, Problem 1 gives an example of a function that is not identically zero but whose RS integral is zero; in particular, its Riemann integral, taking $\alpha(x) = x$, is zero. Second, if we took a RS integral with an α that is constant on an interval, then a function could satisfy $f \geq 0$ and $\int f\, d\alpha = 0$ but could be nonzero somewhere on the interval where α is constant.

3. Rudin, Chapter 6 #3

SOLUTION Let's specify a partition P with five points:

$$-1 = t_0 < t_1 < 0 = t_2 < t_3 < t_4 = 1\,.$$

For all three functions β_i, we have

$$U(P,f,\beta_i) - L(P,f,\beta_i) = \sum_2^3 (M_j - m_j)(\beta_i(t_{j-1}) - \beta(t_j))\,,$$

where

$$M_j = \sup_{[t_{j-1},t_j]} f(t)\,, \quad m_j = \inf_{[t_{j-1},t_j]} f(t)\,,$$

since $\beta_i(t_1) - \beta_i(t_0) = 0$ and $\beta_i(t_4) - \beta_i(t_3) = 0$ for all the β_i. Now look at each case.

(a) For β_1, we also have $\beta_1(t_2) - \beta_1(t_1) = 0$, so in this case

$$U(P,f,\beta_1) - L(P,f,\beta_1) = (M_3 - m_3)(\beta_i(t_3) - \beta(t_2)) = M_3 - m_3\,.$$

Now, if $f(0+) = f(0)$ then for any $\varepsilon > 0$ we have a $\delta > 0$ such that $|f(t) - f(0)| < \varepsilon/2$ on $[0,\delta]$ and so we can certainly find a partition such that $M_3 - m_3 < \varepsilon$, so then $f \in \mathcal{R}(\beta_1)$ by Theorem 6.6. Conversely if $f \in \mathcal{R}(\beta_1)$ then for each $\varepsilon > 0$ such a partition exists. But if $M_3 - m_3 < \varepsilon$ then $\lim_{t\to 0} f(t) = t(0)$ as t decreases to 0, so $f(0+) = f(0)$. Furthermore in this case we have $M_3 \to f(0)$ and $m_3 \to f(0)$ so $\int f\,d\beta_1 = f(0)$.

(b) For β_2, the picture is similar, except that now $\beta_2(t_3) - \beta_2(t_2) = 0$ and the non-zero summand is $M_2 - m_2$. Then the result is

> A function $f \in \mathcal{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and then $\int f\,d\beta_2 = f(0)$.

This is proved the same way.

(c) When we look at β_3, we retain both terms, but with a factor of 1/2:

$$U(P,f,\beta_3) - L(P,f,\beta_3) = \frac{1}{2}\Big((M_2 - m_2) + (M_3 - m_3)\Big)\,.$$

Now, $M_i - m_i \geq 0$, so the only way this can be made arbitrarily small is for the conditions in both part (a) and part (b) to hold. Conversely, if both hold then $f \in \mathcal{R}(\beta_3)$. But if both hold, then $f(0+) = f(0) = f(0-)$ so f is continuous at 0.

(d) It also follows from part (c) that if f is continuous at 0 then $M_i \to f(0)$ for $i = 2$ and $i = 3$, so we can calculate the limit of $U(P, f, \alpha)$ and it is also $f(0)$. So we conclude that if f is continuous at 0 then the common value of all three integrals is $f(0)$.

4. Rudin, Chapter 6 #8

SOLUTION We have the definition of $\int_a^\infty f$ as the limit of $\int_a^b f$, so we start by noting some upper and lower sums for the finite integral. (Since f is monotonic, then by Theorem 6.9 $f \in \mathcal{R}$ on any finite interval $[1, b]$.) Taking a partition $P = \{1, 2, 3, \ldots N, b\}$ (where N is the greatest integer less than b) we have

$$L(P, f) = \sum_2^N f(n) + f(b)(b-N) \leq \int_1^b f \leq \sum_1^{N-1} f(n) + f(N)(b-N) = U(P, f).$$

Now, if $\lim_{b\to\infty} \int_1^b f$ exists, then letting $b \to \infty$ on the left we see that $\sum_2^\infty f(n)$ is finite (so the sum converges). Conversely, if the sum converges then the sum on the right side is finite (since $f(N) \leq f(1)$ is finite and $b - N < 1$). Hence $\int_1^b f$ is bounded as $b \to \infty$. But since $f \geq 0$, the integral is an increasing function of b. Thus, as an increasing function that is bounded above, it has a limit as $b \to \infty$.

5. Rudin, Chapter 6 #16

SOLUTION Questions on the zeta function. We note that $\zeta(s) = \sum \frac{1}{n^s}$ converges when $s > 1$.

(a) Begin with the right side:

$$s \int_1^\infty \frac{[x]}{x^{s+1}}\, dx = s \lim_{N\to\infty} \int_1^N \frac{[x]}{x^{s+1}}\, dx = \lim_{N\to\infty} s \sum_1^{N-1} \int_k^{k+1} \frac{[x]}{x^{s+1}}\, dx\,,$$

where the existence of the integral is guaranteed by Problem 8. Now

$$\int_k^{k+1} \frac{[x]}{x^{s+1}}\, dx = k \int_k^{k+1} \frac{1}{x^{s+1}}\, dx = -\left. \frac{k}{s} \frac{1}{x^s} \right]_k^{k+1} = \frac{k}{s} \left(\frac{1}{k^s} - \frac{1}{(k+1)^s} \right).$$

Next we put this expression into the finite sum:

$$s \sum_1^{N-1} \int_k^{k+1} \frac{[x]}{x^{s+1}}\, dx = s \sum_1^{N-1} \frac{k}{s} \left(\frac{1}{k^s} - \frac{1}{(k+1)^s} \right) = \sum_1^{N-1} \frac{k}{k^s} - \sum_1^{N-1} \frac{k}{(k+1)^s}$$

and shift the indices in the second sum to get

$$\sum_1^{N-1} \frac{k}{k^s} - \sum_2^N \frac{k-1}{k^s} = \frac{1}{1^s} + \sum_2^{N-1} \frac{1}{k^s} - \frac{N-1}{N^s}.$$

Now let $N \to \infty$. The first term can be incorporated into the sum, and the final term tends to zero, so we get $\zeta(s)$, as required.

(b) First we establish the identity, assuming that $s > 1$. In this case, we know that

$$s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx = \zeta(x)\,,$$

and so we need to show that

$$s\int_1^\infty \frac{x}{x^{s+1}}\,dx = s\int_1^\infty \frac{1}{x^s}\,dx = \frac{s}{s-1}\,.$$

Still assuming that $s > 1$, we know that the last integral is the limit of a finite integral, from 1 to b, say. But this can be evaluated using the fundamental theorem of calculus (Theorem 6.21), since the derivative of

$$-\frac{1}{(s-1)x^{s-1}}$$

is the integrand ($1/x^s$). So we get

$$s\int_1^\infty \frac{1}{x^s}\,dx = \lim_{b\to\infty} s\int_1^b \frac{1}{x^s}\,dx = -\lim_{b\to\infty} \frac{s}{s-1}\,\frac{1}{x^s}\bigg]_1^b = \frac{s}{s-1}\,.$$

Now, if $s > 0$, we can test

$$\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx$$

for convergence by using the integral test, Problem 8. For that we note that the numerator of the integrand is between 0 and 1 and hence this integral converges as long as $s + 1 > 1$, or $s > 0$.

Part (b) gives a formula that can be used to extend the definition of $\zeta(s)$ from $s > 1$ to $0 < s < 1$. Notice that even though the integral in (b) converges for $s = 1$, as we just proved, this does not mean that $\zeta(1)$ is defined. It's clear from the original formula for ζ as a sum that $\zeta(s) \to \infty$ as $s \to 1$, and that's clear also from the formula in part (b), at least for s decreasing to 1.

6. Bergman 6.3:0

SOLUTION This is false. There are many counterexamples, such as the indicator function for the rationals, or a bounded function that is discontinuous at a point where α is discontinuous. Put most simply, since any constant function $g(x) \equiv M$ is in $\mathcal{R}(\alpha)$ on $[a, b]$ for any admissible α, if the statement were true then any function f with $|f(x)| \le M$ would be in $\mathcal{R}(\alpha)$ on $[a, b]$, and we know this is not true.