# MATH 5201 ANALYSIS FALL 2015

Homework Assignment #3 September 17 Solutions

1. Rudin, Chapter 2 #12

   SOLUTION We have $K = \{1, \frac{1}{2}, \dots \frac{1}{n}, \dots 0\}$. To prove that $K$ is compact from the definition, we must suppose that we are given an open cover $\{G_\alpha\}$: a collection, that might be uncountable, of open sets such that $K \subset \cup_\alpha G_\alpha$. From this collection, we must be able to select a finite subcollection that covers $K$.

   Focus on the point $0 \in K$. This point lies in some set in the collection, call it $G_1$. Now, since $G_1$ is open, it contains a neighborhood of 0: For some $r > 0$, all points $x \in \mathbb{R}$ with $|x| < r$ are in $G_1$ and so all points $1/n$ with $n > 1/r$ are in $G_1$. That leaves only a finite number of points, and each one is contained in some $G_i$, $2 \le i \le N$, say. Thus, the union of $G_1$ and $G_2, \dots, G_N$ is a finite subcollection that covers $K$. Since we had allowed $\{G_\alpha\}$ to be an arbitrary collection, we have showed that $K$ is compact.

2. Rudin, Chapter 2 #14

   SOLUTION Consider the open sets $G_1 = (1, \frac{1}{3})$ and $G_n = (\frac{1}{n^2}, \frac{1}{n})$, for $n \ge 2$. (Since $n^2 > n$ for $n \ge 2$, these sets are non-empty.) Since for every value $x > 0$ there is an $N$ such that $\frac{1}{N^2} < x$, the countable collection $\{G_n\}$ covers $(0, 1)$. On the other hand, for any finite subcollection, there is a largest value, say $N$, in the subcollection, and the point $\frac{1}{N^2}$, which is in $(0, 1)$, will not be in the subcollection. Thus, no finite subcollection can cover $(0, 1)$.

3. Rudin, Chapter 2 #16

   SOLUTION This question can be answered by direct construction, ignoring the fact that $\mathbb{Q} \subset \mathbb{R}$. However, since $\mathbb{Q} \subset \mathbb{R}$, it is efficient to use Theorems 2.30 and 2.33 in Rudin and avoid detailed constructions.

   Specifically, if we define $F = \{x \in \mathbb{Q} \mid 2 \le x^2 \le 3\}$ then $E = F$, and if we define $G = (-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})$ in $\mathbb{R}$ then $E = G \cap \mathbb{Q}$.

   Then Theorem 2.30 says that $E$ is open in $\mathbb{Q}$, since $G$ is open in $\mathbb{R}$. On the other hand, $E^c$ is also open, since $E^c = F^c$ and $F^c$ is the intersection with $\mathbb{Q}$ of an open set $(-\infty, -\sqrt{3}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{3}, \infty)$.

   So, $E$ is both open and closed in $\mathbb{Q}$.

The set $E$ is also bounded in $\mathbb{Q}$, since $x \in E$ implies $|x| < \sqrt{3}$.

Finally, $E$ is not compact, by Theorem 2.33: $E \subset \mathbb{Q} \subset \mathbb{R}$ and $E$ is not a closed subset of $\mathbb{R}$ (since the endpoints $\pm\sqrt{3}, \pm\sqrt{2}$ are limit points of $E$ which are not in $E$) and hence is not compact in $\mathbb{R}$. Thus, by Theorem 2.33, $E$ is not compact in $\mathbb{Q}$.

4. Bergman 2.3:1 – Show that if $E_1, \ldots, E_n$ are compact subsets of a metric space $X$, then their union $F \equiv E_1 \cup \ldots \cup E_n$ is also compact.

   Solution Let $\{G_\alpha\}$ be an open cover of $F$. For each $i$, $1 \leq i \leq n$, there is a subcollection, $\{G_\alpha^{(i)\}}$ that covers $E_i$. (Some sets may appear in more than one subcollection, but none appears more than $n$ times.) Since each $E_i$ is compact, we can select a finite subcover, say $G_1^{(i)}, \ldots, G_{\alpha_i}^{(i)}$ of $\alpha_i$ sets that covers $E_i$. Then the union of all these sets covers $F$, and this union contains at most $\alpha_1 + \ldots + \alpha_n$ sets. (The " at most" is needed because it may be that not all the sets are distinct.)

5. Bergman 2.3:3 – Suppose $K$ is a compact metric space and $\varepsilon$ a positive real number. Show that there is a positive integer $N$ such that every set of $N$ points of $K$ includes at least two points of distance $< \varepsilon$ apart. (Hint: Take $N$ to be greater than the number of sets in some covering of $K$ by neighborhoods of radius $\varepsilon/2$.)

   Solution Following the hint, we can form a covering $\{G_\alpha\}$ of $K$ by letting the index $\alpha = x$ run over all the points $x$ of $K$ and letting $G_x = N_r(x)$; that is, a neighborhood of $x$ of radius $r$, where we choose $r < \varepsilon/2$. Since $K$ is compact, there is a finite subcover, $\{G_1, \ldots, G_N\}$ containing $N$ sets. Now, at this point we must assume that $K$ contains more than $N$ points. Otherwise there is nothing else to say. Furthermore, since we would like to allow $\varepsilon$ to be arbitrary, we must assume that $K$ has an infinite number of points. In that case, at least one of the $G_1$ must have an infinite number of points. In particular, that set will have two points, and they are a distance $< \varepsilon$ apart.