Math 5201 Analysis Fall 2015

Homework Assignment #5 October 1 Solutions

1. Rudin, Chapter 3 #20

   Solution We need to show $\{p_n\}$ converges to p. So suppose we are given an $\varepsilon > 0$. Since $\{p_{n_k}\}$ converges to $p \in X$, there is a K such that for $k > K$ we have $d(p_{n_k}, p) < \varepsilon/2$. Since $\{p_n\}$ is Cauchy, there is an N such that when $n, m > N$ then $d(p_n, p_m) < \varepsilon/2$.

   Let $M = \max\{n_K, N\}$ and let n_k be a number in the subsequence such that $k > K$ and $n_k > M$.

   Then if $n > M$ we have

   $$d(p_n, p) \le d(p_n, p_{n_k}) + d(p_{n_k}, p) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

   Hence the limit of $\{p_n\}$ is p.

2. Rudin, Chapter 3 #22 (Baire's Theorem)

   Solution We are given a complete metric space X and a collection G_n of open, dense subsets of X. Since this is a little difficult to imagine, here's a simple example. Suppose $X = \mathbb{R}$ and let G_n be all the real numbers except rational numbers with denominator n. Since $X \backslash G_n$ consists of isolated points, it's clear that G_n is dense and open. (On the other hand, $\cap G_n$ consists of $\mathbb{R}$ with all the rational numbers removed – that is, $\cap G_n$ is all the irrational numbers. This set is non-empty and in fact dense, but it is not an open set.)

   So, for our problem, start with any open neighborhood, say $N_{r_1}(x_1)$ in G_1, with radius $r_1 > 0$, and let $E_1 = \overline{N_{r_1/2}(x_1)}$. We construct the closed sets E_n recursively. Once we have E_{n-1}, a closed set of the form $\overline{N_{r_{n-1}/2}(x_{n-1})}$ we observe that the set $A_n = N_{r_{n-1}/2}(x_{n-1}) \cap G_n \subset E_{n-1}$ is open (intersection of two open sets) and nonempty (since G_n is dense it must contain points of any open subset of X), so there is a point x_n in A_n, and since A_n is open there is a neighborhood $N_{r_n}(X_n) \subset A_n$. Furthermore, $r_n \le r_{n-1}/2$ (the radius of E_{n-1}) and we can set

   $$E_n = \overline{N_{r_n/2}(x_n)}$$

for the next nested, closed set in the sequence. Since $r_n \le r_{n-1}/2$ for all n, the diameters of the sets tend to zero as $n \to \infty$, so we can apply the conclusion of Problem 21, as suggested by the hint.

To see that $\cap G_n$ is dense is not much more difficult (though not required in the problem): We began with a point $x_1 \in X$ and any neighborhood of x_1. The intersection point we found is in this neighborhood. Thus, any neighborhood of any point in X contains a point of $\cap G_n$.

3. Rudin, Chapter 3#23

SOLUTION If we use the hint, then we have, immediately,

$$d(p_n, q_n) - d(p_m, q_m) \le d(p_n, p_m) + d(q_m, q_n)\,,$$

for any n and m; and rewriting this with n and m interchanged we also have

$$d(p_m, q_m) - d(p_n, q_n) \le d(p_n, p_m) + d(q_m, q_n)\,,$$

and so

$$|d(p_n, q_n) - d(p_m, q_m)| \le d(p_n, p_m) + d(q_m, q_n)\,,$$

for any n and m. (Note that while p_n and q_n are in a metric space with metric d, the sequence $\{d(p_n, q_n)\}$ is a sequence of real numbers.) Now suppose we are given any $\varepsilon > 0$. Since $\{p_n\}$ and $\{q_n\}$ are Cauchy, we can find N_1 and N_2 such that if $n, m > N_1$ then

$$d(p_n, p_m) < \frac{\varepsilon}{2}$$

and if $n, m > N_2$ then

$$d(q_n, q_m) < \frac{\varepsilon}{2}\,.$$

Then if $n, m > \max\{N_1, N_2\}$ we will have

$$|d(p_n, q_n) - d(p_m, q_m)| \le d(p_n, p_m) + d(q_m, q_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

so the sequence $\{d(p_n, q_n)\}$ is Cauchy in $\mathbb{R}$, and since $\mathbb{R}$ is a complete metric space, then the sequence converges.

4. Bergman 3.2:2 (Find four sequences in $\mathbb{R}$ whose subsequential limit points are (a) $\emptyset$ (b) $\mathbb{Z}$ (c) $[0, 1]$ (d) $\mathbb{R}$.)

SOLUTION

(a) Any sequence of isolated points, for example $s_n = n$.

(b) If we take any sequence that converges to 0, say $a_n = 1/n$, then the indexed sequence $s_{nk} = k + a_n$ will converge to k, for a fixed $k \geq 0$, and $t_{nk} = -k + a_n$ for $k > 0$ will converge to $-k$. Combine these two countable collections of sequences into a single sequence in some reasonable way; for example we could take

$$p_1 = s_{10}\,, \quad p_2 = s_{11}\,, \quad p_3 = t_{11}\,, \quad p_4 = s_{12}\,, \quad p_5 = s_{21}\,, \quad p_6 = t_{12}\,,$$

and continue through the s's and t's with $n+k$ increasing. Then every integer is the limit of a subsequence of $\{p_n\}$, (constructed by taking what is possibly a subsequence of s_{nk} or t_{nk} to get n or $-n$), and there are no other subsequential limits since no other points are limit points of the set of values $\{p_n\}$.

(c) The standard enumeration of the rationals in $[0, 1]$ is to order $r = p/q$ by increasing values of $p + q$, and within that by increasing p (say). Since the rationals are dense in $[0, 1]$ there are subsequences that converge to every real number in $[0, 1]$. Specifically, for a given x in $[0, 1]$ choose the number in each group with $p + q = N$ for a given N that is closest to x; as $n \to \infty$ this sequence tends to x. (If x is rational, then x appears as a point in the sequence; further points in the sequence will differ from x but still tend to x.)

(d) Once we have the sequence $\{p_n\}$ in part (c), then the sequences $\{k + p_n = q_{kn}\}$ and $\{-k + p_n = s_{kn}\}$ will have subsequences converging to sets $[\pm k, \pm k + 1]$. Combine these doubly indexed sets as in part (b).