Math 5201: Suggested Solution for HW Set 6

Rudin

#6 Let $s_n = \sum_{k=1}^n a_k$.

(a) The partial sum $s_n = \sqrt{n+1} - \sqrt{1} \to +\infty$. Hence $\sum a_n$ diverges.

(b) We estimate a_n from above by

$$0 \le a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \le \frac{1}{n^{3/2}}.$$

Since $3/2 > 1$, the series $\sum \frac{1}{n^{3/2}}$ converges (Thm 3.28). Hence $\sum a_n$ converges by comparison test.

(c) Since $\sqrt[n]{|a_n|} = \sqrt[n]{n} - 1 \to 0$ as $n \to \infty$, we deduce by root test that $\sum a_n$ converges.

(d) For $|z| \le 1$, the terms $a_n = \frac{1}{1+z^n}$ is either undefined, or satisfies

$$|a_n| \ge \frac{1}{1+|z|^n} \ge \frac{1}{2} > 0.$$

Hence $a_n \not\to 0$ and $\sum a_n$ does not converge.

For $|z| > 1$, choose $\delta > 0$ so that $(1-\delta)|z| > 1$. This implies that for all $n \ge 1$,

$$(1-\delta)|z|^n > 1,\ \forall n \ge 1 \iff |z|^n - 1 > \delta|z|^n,\ \forall n \ge 1.$$

Hence $|a_n| \le \frac{1}{|z|^n - 1} \le \delta^{-1}\frac{1}{|z|^n}$. Since $\sum \frac{1}{|z|^n}$ converges as a geometric series, $\sum a_n$ converges by comparison test.

#11

(a) Case $a_n \to 0$. Then $\{a_n\}$ is bounded. i.e. for some $M > 0$, $|a_n| \le M$ for all n. Then $\frac{a_n}{1+a_n} \ge \frac{a_n}{1+M} \ge 0$. And $\sum \frac{a_n}{1+a_n}$ diverges by comparison test. Case $a_n \not\to 0$. Since a_n is non-negative, there exists $\epsilon_0 > 0$ such that $a_n \ge \epsilon_0$ for infinitely many n. Hence

$$\frac{a_n}{1+a_n} - \frac{\epsilon_0}{1+\epsilon_0} = \frac{a_n - \epsilon_0}{(1+a_n)(1+\epsilon_0)} \ge 0 \quad \text{for infinitely many } n.$$

This implies that $\frac{a_n}{1+a_n} \not\to 0$ and hence $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Using the fact that $a_n \ge 0$, $s_{N+i} \le s_{N+k}$ for all $i = 1, ..., k-1$, so that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \ge \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Since $\sum a_n$ is a divergent series of non-negative terms, $s_n \to +\infty$. For each fixed N, there exists k such that $1 - \frac{s_N}{s_{N+k}} \geq \frac{1}{2}$. Hence, for each N, there exists $k \geq 0$ such that

$$\left| \sum_{n=N+1}^{N+k} \frac{a_n}{s_n} \right| \geq 1 - \frac{s_N}{s_{N+k}} \geq \frac{1}{2}.$$

This implies, by Cauchy's criterion, that $\sum \frac{a_n}{s_n}$ diverges.

(c) We have, by using $s_n \geq s_{n-1}$ in the last inequality, that

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{a_n}{s_n s_{n-1}} \geq \frac{a_n}{s_n^2}.$$

Let $\epsilon > 0$ be given. Since $\sum a_n$ is a divergent series of non-negative terms, $s_n \to +\infty$. Hence we can choose N such that

$$\frac{1}{s_n} < \frac{\epsilon}{2} \qquad \text{for all } n \geq N-1.$$

This yields, for the N chosen above and every $k \geq 0$,

$$\left| \sum_{n=N}^{N+k} \frac{a_n}{s_n^2} \right| \leq \sum_{n=N}^{N+k} \left(\frac{1}{s_{n-1}} + \frac{1}{s_n} \right) = \frac{1}{s_{N-1}} - \frac{1}{s_{N+k}} < \epsilon.$$

The Cauchy's criterion implies that $\sum \frac{a_n}{s_n^2}$ converges.

(d) For $\sum \frac{a_n}{1+na_n}$: It depends: (Thanks to ≈ 8 students who gave the correct solution!)

If $a_n = \frac{1}{n}$, then both $\sum a_n$ and $\sum \frac{a_n}{1+na_n}$ diverges. If

$$a_n = \begin{cases} 1 & \text{for } n = k^2 \text{ for some k,} \\ 2^{-n} & \text{otherwise,} \end{cases}$$

then $\sum a_n$ diverges but $\sum \frac{a_n}{1+na_n}$ converges as

$$\sum \frac{a_n}{1+na_n} \leq \sum_k \frac{1}{1+k^2} + \sum_{n \neq k^2} 2^{-n}.$$

However, the statement can be true in general if we suppose in addition that $\{a_n\}$ is monotone decreasing, then $\{\frac{a_n}{1+na_n}\}$ is also decreasing, and

$$\begin{aligned} \sum \frac{a_n}{1+na_n} \text{ converges} \quad &\Longrightarrow \quad \sum \frac{2^k a_{2^k}}{1+2^k a_{2^k}} \text{ converges (by Theorem 3.27)} \\ &\Longrightarrow \quad \sum 2^k a_{2^k} \text{ converges (by part (a))} \\ &\Longrightarrow \quad \sum a_n \text{ converges (by Theorem 3.27)} \end{aligned}$$

i.e. $\sum a_n$ diverges as a non-negative, decreasing sequence implies that $\sum \frac{a_n}{1+na_n}$ diverges.

For $\sum \frac{a_n}{1+n^2a_n}$: $\frac{a_n}{1+n^2a_n} \le \frac{1}{n^2}$ so it always converges, by comparison test.

#17 From the definition

$$x_1 > \sqrt{\alpha}, \quad x_{n+1} = \frac{\alpha + x_n}{1 + x_n}, \tag{1}$$

we deduce from definition that $x_2 > 0$ and

$$x_2 - \sqrt{\alpha} = \frac{\alpha + x_1 - \sqrt{\alpha} - \sqrt{\alpha}x_1}{1 + x_1} = \frac{\sqrt{\alpha} - 1}{1 + x_1}(\sqrt{\alpha} - x_1) < 0, \tag{2}$$

and

$$x_{n+2} = \frac{\alpha + \frac{\alpha + x_n}{1+x_n}}{1 + \frac{\alpha+x_n}{1+x_n}} = \frac{2\alpha + (1+\alpha)x_n}{1 + \alpha + 2x_n}. \tag{3}$$

Hence

$$x_{n+2} - x_n = \frac{2\alpha + (1+\alpha)x_n - (1+\alpha)x_n - 2x_n^2}{1 + \alpha + 2x_n} = \frac{2}{1 + \alpha + 2x_n}(\alpha - x_n^2). \tag{4}$$

(a)

Claim 1. *For n odd, $x_n > \sqrt{\alpha}$ and $x_{n+1} < x_n$.*

We prove by induction. The first case is $n = 1$, and we have $x_1 > \sqrt{\alpha}$ by definition, and by (4),

$$x_3 - x_1 = \frac{2}{1 + \alpha + 2x_1}(\alpha - x_1^2) < 0.$$

Suppose the claim holds for some odd n. Then from (3),

$$x_{n+2} - \sqrt{\alpha} = \frac{2\alpha + (1+\alpha)x_n}{1 + \alpha + 2x_n} - \sqrt{\alpha} = \frac{(1+\sqrt{\alpha})^2(x_n - \sqrt{\alpha})}{1 + \alpha + 2x_n}$$

By induction assumption, $x_n > \sqrt{\alpha}$, so we deduce $x_{n+2} > \sqrt{\alpha}$. We also have, $x_{n+2} < x_n$ from (4).

(b) We have already shown that $0 < x_2 < \sqrt{\alpha}$. We can (similarly as (a)) show by induction that $0 < x_2 < x_4 < ... < \sqrt{\alpha}$.

(c) Since $\{x_{2n}\}$ and $\{x_{2n-1}\}$ are bounded and monotone sequences,

$$0 < x_2 \le \beta := \lim_{n\to\infty} x_{2n} \le \sqrt{\alpha}, \quad \gamma := \lim_{n\to\infty} x_{2n-1} \ge \sqrt{\alpha}$$

both converges. From (3), we have

$$x_{2n+2} = \frac{2\alpha + (1+\alpha)x_{2n}}{1+\alpha+2x_{2n}}.$$

Letting $n \to \infty$, we have

$$\beta = \frac{2\alpha + (1+\alpha)\beta}{1+\alpha+2\beta} \quad \Longleftarrow \quad \beta = \sqrt{\alpha}.$$

Similarly, taking $n = 2k$ to be even in (1), and letting $k \to \infty$, we deduce that

$$\gamma = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} \frac{\alpha + x_{2k}}{1 + x_{2k}} = \frac{\alpha + \sqrt{\alpha}}{1+\sqrt{\alpha}} = \sqrt{\alpha}.$$

Hence the set of subsequential limits of $\{x_n\}$ is a singleton set $\{\sqrt{\alpha}\}$ and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

(d)

$$\left|x_{n+1} - \sqrt{\alpha}\right| = \frac{\sqrt{\alpha}-1}{1+x_n}|x_n - \sqrt{\alpha}|$$

Since $x_n \to \sqrt{\alpha}$, we see that $\frac{\sqrt{\alpha}-1}{1+x_n} \to \frac{\sqrt{\alpha}-1}{1+\sqrt{\alpha}} < 1$. i.e.

$$\left|x_{n+1} - \sqrt{\alpha}\right| \approx C|x_n - \sqrt{\alpha}|$$

for some $C < 1$ for all n sufficiently large. This convergenece is slower than that of #16.

Bergman 3.9 : 0

(a) True. $\sum a_n$ converges by ratio test, since $\limsup |a_{n+1}/a_n| = 1/2 < 1$.

(b) False. Consider for instance

$$a_n = \begin{cases} 1/k^2 & \text{for } n = 2k, \\ 2/k^2 & \text{for } n = 2k+1. \end{cases}$$

Then

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{k^2}{2(k+1)^2} & \text{for } n = 2k, \\ 2 & \text{for } n = 2k+1, \end{cases}$$

so that $\limsup |a_{n+1}/a_n| = 2 > 1$ but

$$\sum a_n = \sum \left(\frac{1}{k^2} + \frac{2}{k^2}\right)$$

is convergent.