Math 5201 Analysis Fall 2015

Homework Assignment #7 October 14 Solutions

1. Rudin, Chapter 4 #1 (Suppose that f satisfies $\lim_{h\to 0}[f(x+h)-f(x-h)] = 0$ for every $x \in \mathbb{R}$. Does this imply f is continuous?

   Solution No. For example, take any function that is continuous on $\mathbb{R}$ and change its value at a single point, or on any collection of points that is nowhere dense, for example, on the Cantor set or the integers. It's clear that if f is continuous at a point x_0 then

   $$\lim_{h\to 0}[f(x_0+h)-f(x_0-h)] = \lim_{h\to 0}[f(x_0+h)-f(x_0)]+[f(x_0)-f(x_0-h)] = 0$$

   so the condition is necessary. So a function like

   $$f(x) = \begin{cases} 1, & x \in C, \\ 0, & x \notin C, \end{cases}$$

   (where C is the Cantor set) satisfies the given condition if $x \notin C$, because the complement of C is open and so every such point has a neighborhood where $f(x) = 0$. But if $x \in C$ then, for sufficiently small h, $x+h$ and $x-h$ are both in the complement of C (otherwise C would be dense at x), and so the given condition also holds. (Most students took the exceptional set to be a single point, and that's fine, but it's interesting that the function can fail to be continuous at quite a large number of point.)

2. Bergman #3.10:0 (Say whether the following statement is true or false: If a power series $\sum c_n z^n$ converges at $z = 1+3i$, then it also converges at $z = 2+2i$.)

   Solution Yes, this is true. According to Theorem 3.39 every power series of the form $\sum c_n z^n$ has a radius of convergence and it converges for all complex numbers inside its radius of convergence, and diverges for all complex numbers outside it. In this case, since it converges for $1+3i$, the radius of convergence must be as large as $|1+3i| = \sqrt{1+3^2} = \sqrt{10}$, and since $|2+2i| = \sqrt{8} < \sqrt{10}$ Theorem 3.39 tells us that it converges at $2+2i$.

3. Bergman #3.12:2 Let $\sum a_n z^n$ be a power series with radius of convergence $R > 0$.

(a) Show that for all complex numbers z with $|z| < R$, the given series converges absolutely.

(b) Show by three examples that for a complex number z with $|z| = R$, the series $\sum a_n z^n$ may diverge, may converge nonabsolutely, or may converge absolutely. (In fact, you can find a power series that shows two of the above phenomena at different values of z with $|z| = R$; but an example of the remaining phenomenon requires a different power series. Do you see why?)

SOLUTION

(a) Recall the the radius of convergence is $R = 1/\alpha$ where $\alpha = \overline{\lim}|a_n|^{1/n}$. Now, for any $z \in \mathbb{C}$, look at $\sum |a_n z^n| = \sum |a_n||z|^n$. The series converges absolutely if this series converges. Compare with the geometric series $\sum (\alpha|z|)^n$, which converges if $\alpha|z| < 1$. So, if $|z| < R$, write $|z| = R - \tau/\alpha$, with $\tau > 0$. Now, for all but a finite number of terms, $|a_n|^{1/n} \leq \alpha(1+\tau/2)$, so for sufficiently large n, $|\alpha|^{1/n}|z| < 1-\tau/2 < 1$, so $\sum |a_n z^n|$ converges by comparison with a covergent geometric series.

(b) The series $\sum z^n/n^2$ has $R = 1$ and converges absolutely when $|z| = 1$. The series $\sum nz^n$ also has $R = 1$ and diverges when $|z| = 1$. The series $\sum z^n/n$ has $R = 1$ and converges conditionally when $z = -1$; it diverges when $z = 1$. If a series converges absolutely for one value of z with $|z| = R$, then it will converge for all z with $|z| = R$, since absolute convergence is determined by $|z|$. Thus, the example for absolute convergence cannot be combined with either of the other two.

4. Bergman #3.13:0 (Say whether the following statement is true or false: If $\sum a_n = A$ and $\sum b_n = B$, and these series converge absolutely, then $\sum a_n b_n = AB$.)

SOLUTION False. There is no reason for this to be true, and here's an example. Take two convergent geometric series,

$$\sum r^n \quad \text{and} \quad \sum s^n .$$

Then (assuming all sums start at $n = 0$)

$$\sum r^n s^n = \frac{1}{1-rs} ,$$

but

$$\left(\sum r^n\right)\left(\sum s^n\right) = \frac{1}{1-r}\frac{1}{1-s} = \frac{1}{1-r-s+rs}$$

which is a different number in general.

5. Bergman #3.13:1 (Radii of convergence of sum and product series. Suppose $\sum a_n$ and $\sum b_n$ are series, and let their sum in the sense of Theorem 3.47 and their product in the sense of Definition 3.48 be denoted $\sum s_n$ and $\sum p_n$ respectively.

   (a) Show that if r is a real number such that $\sum a_n z^n$ and $\sum b_n z^n$ both have radius of convergence $\geq r$, then $\sum s_n z^n$ and $\sum p_n z^n$ also have radii of convergence $\geq r$. (Suggestion: Don't use the "lim sup" formula for the radius of convergence, but its characterization in terms of where power series converge and where they diverge, together with the result of 3.12:2(a).)

   (b) Deduce from (a) that if $\sum a_n z^n$ and $\sum b_n z^n$ have different radii of convergence, then the radius of convergence of $\sum s_n z^n$ is equal to the smaller of those two radii.

   (c) Also deduce from (a) that if the radius of convergence of $\sum p_n z^n$ is less than that of $\sum a_n z^n$, then the radius of convergence of $\sum b_n z^n$ is $\leq$ that of $\sum p_n z^n$.

   Solution

   (a) Suppose $|z| < r$. Since we do not know the series converge, we should consider partial sums: Then

   $$|\sum_1^N s_n z^n| \leq \sum_1^N |a_n z^n| + \sum_1^N |b_n z^n|$$

   and both these series converge (with problem 3.12:2(a) the reference for absolute convergence of the series). Hence, since the sum of convergent series converges, we have convergence of $\sum s_n z^n$.
   From Definition 3.48,

   $$\sum p_n z^n = (\sum a_n z^n)(\sum b_n z^n)$$

   (if we could take the infinite sum) and if $|z| < r$ the right side converges; by taking limits over partial sums, we see the left side converges also.

   (b) On the other hand, suppose that $|z| > r$ where r is the smaller radius of convergence (say it's the radius of convergence for $\sum a_n z^n$). Then write

   $$\sum a_n z^n = \sum s_n z^n - \sum b_n z^n .$$

   Now, if $\sum s_n z^n$ converged, then so would the right side of this equation, by part (a), and then so would the left side, but a series cannot converge if $|z|$ is larger than its radius of convergence, so the left side does not converge, and so we conclude that $\sum s_n z^n$ does not converge.

(c) At any value of z for which $\sum a_n z^n \neq 0$, we can write

$$\sum b_n z^n = \frac{\sum p_n z^n}{\sum a_n z^n}.$$

Now suppose z is a value where $\sum a_n z^n$ converges but $\sum p_n z^n$ diverges, then the right side of this equation diverges, and hence so does the left, and z is also a value where $\sum b_n z^n$ diverges, so if $|z| > R_p$ where R_p is the radius of convergence of $\sum p_n z^n$, then $|z| \geq R_b$, where R_b is the radius of convergence of $\sum b_n z^n$. For completeness, note that if $\sum a_n z^n = 0$ for all z, then all coefficients of $\sum a_n z^n$ are zero, and so are all coefficients of $\sum p_n z^n$. In that case, both $\sum a_n z^n$ and $\sum p_n z^n$ have infinite radius of convergence, so the conclusion also holds, somewhat vacuously.

6. Bergman #3.14:0 (Say whether each of the following statements is true or false.

   (a) There is a rearrangement of the series $\sum(-1)^n n^{-1}$ which converges to 2011.(T)
   (b) If a series $\sum a_n$ has the property that all of its rearrangements converge, it is absolutely convergent.(T)
   (c) If a series $\sum a_n$ has the property that some rearrangement converges, then it itself converges.(F)

   SOLUTION

   (a) This is true, and follows from Theorem 3.54 by taking $\alpha = \beta = -2011$ there.
   (b) This is also true. Again Theorem 3.54 implies this result, for if the series converged only conditionally then we could take, say, $\beta = \infty$ (note that this is allowed in the theorem), and find a rearrangement that diverges. Since we cannot find such a rearrangement, the series must converge absolutely.
   (c) No, this is false. Since, again by Theorem 3.54, given any conditionally convergent series $\sum a_n$, there is a rearrangement $\sum a'_n$ that diverges (take $\beta = \infty$), then we can take $\sum a'_n$ to be our "given" series and it diverges, but its rearrangement $\sum a_n$ converges. (That is, we just switch the labels.)

7. Bergman #3.14:4 (Find a series which converges by the root test, and a rearrangement of this series for which the root test gives no information. However, note a reason why, in a situation of this sort, the rearranged series must still converge.)

SOLUTION The rearranged series will converge, because any series that passes the root rest is absolutely convergent, and Theorem 3.55 says that in that case all the rearrangements also converge.

We can create an example by starting with a geometric series, $\sum r^n$ with $0 < r < 1$. Now, for this series,

$$\overline{\lim} |a_n|^{1/n} = r < 1\,,$$

so the root test shows convergence. Take $r = \frac{1}{2}$ for concreteness. We can create a rearrangement with $\overline{\lim} |a_n|^{1/n} = 1$ by moving a few terms early in the series to places much later (and replacing them with the later values). To do this in an algorithmic way, take the term in the $10k$ place (which is 2^{-10k}) and exchange it with the term in the 2^{10k} place, which is

$$\frac{1}{2^{2^{10k}}}$$

(but we don't care about its value). Do this for $k = 1, 2, \ldots$. Notice that no terms ever get moved more than once, because we are always taking a term in a position of the form $10k$ – so divisible by 10 – and moving it to a position of the form 2^N – so, a power of 2 and not divisible by 10. So once we've made the k-th switch, we never visit that term again. Thus we can calculate, for this collection of rearranged terms, that the term in the position 2^{10k} is $1/2^{10k}$. That is, there is a subsequence of terms of the form $a_N = 1/N$, for $N = N_k = 2^{10k}$. Now

$$\overline{\lim} |a_N|^{1/N} = \overline{\lim} \left(\frac{1}{N}\right)^{1/N} = 1\,,$$

where $N \to \infty$ as $k \to \infty$. Hence for this rearrangement, the root test gives no information.