MATH 5201 ANALYSIS FALL 2015

Homework Assignment #8 October 28 Solutions

1. Rudin, Chapter 4 #8

   SOLUTION The function f is real and uniformly continuous (u.c.) on $E \subset \mathbb{R}$, and since E is bounded, there are upper and lower bounds: $E \subset [a, b]$, say.

   Since f is u.c., we can take $\varepsilon = 1$ and there is a $\delta > 0$ such that for all $x, y \in E$, $|f(x) - f(y)| < 1$ when $|x - y| < \delta$. Fix an integer $N > 1/\delta$ (using the archimedean property), and partition $[a, b]$ into N subintervals $[a_i, b_i]$ of length $(b - a)/N$. (Here $a_1 = a$, $b_1 = a_2 = a + (b - a)/N$ and so on.)

   For each i, $1 \leq i \leq N$, if $E \cap [a_i, b_i] \neq \emptyset$, choose a point $x_i \in E \cap [a_i, b_i]$ and note that $|f(x) - f(x_i)| < 1$ for all $x \in E \cap [a_i, b_i]$, since $|x - x_i| < \delta$ there. If $E \cap [a_i, b_i] = \emptyset$, ignore that interval.

   Now let $M = \max\{|f(x_i)|, 1 \leq i \leq N\}$ (taking the bound over the intervals where x_i has been defined). This is a finite number, since we are taking a bound over a finite set of points. By construction, for any $x \in E$, $|f(x)| \leq M + 1$, so f is bounded.

   If E is not bounded, the function $f(x) = x$ is u.c. and is not bounded.

2. Rudin, Chapter 4 #10

   SOLUTION Following the suggestion, we assume that f is not u.c. and so there is an $\varepsilon > 0$ and sequences $\{p_n\}, \{q_n\}$ in X with $d_X(p_n, q_n) \to 0$ and $d_Y(f(p_n), f(q_n)) > \varepsilon$.

   Since X is compact, the sequence $\{p_n\}$ has a convergent subsequence, $\{p_{n_k}\}$ with limit p and furthermore the subsequence $\{q_{n_k}\}$ itself has a convergent subsequence with limit q. Relabelling these sequences, we've now constructed a pair of sequences $\{p_j\}$ and $\{q_j\}$ with the properties

   (i) $p_j \to p$ and $q_j \to q$

   (ii) $d_X(p_j, q_j) \to 0$

   (iii) $d_Y(f(p_j), f(q_j)) > \varepsilon$.

   But this results in a contradiction: Since

$$d_X(p, q) \leq d_X(p, p_j) + d_X(p_j, q_j) + d_X(q_j, q)$$

and all three terms on the right side tend to 0 as $j \to \infty$, we must have $p = q$. But f is continuous on X and hence at p, and so for the given ε there is a δ such that for any $r \in X$ with $d_X(p, r) < \delta$ then $d_Y(f(p), f(r)) < \varepsilon/2$. But then for j with $d_X(p, p_j) < \delta$ and $d_X(p, q_j) < \delta$ we have

$$d_Y(f(p_j, q_j)) \leq d_Y(f(p_j), f(p)) + d_Y(f(p), f(q_j)) < \varepsilon$$

which contradicts (iii).

(The text suggests a slightly different version of this argument, by using (iii) to conclude that the sequences $\{p_n\}$ and $\{q_n\}$ cannot have any limit points – or convergent subsequences. It amounts to the same thing.)

3. Rudin, Chapter 4 #14

   Solution We may assume that $f(0) > 0$ and $f(1) < 1$ since otherwise $x = 0$ or $x = 1$ provides a solution. Then if we define $g(x) = f(x) - x$ we know that g is also continuous (by Theorem 4.9 for the sum of two functions and by elementary considerations for the function x or $-x$).

   Since $g(0) > 0 > g(1)$, Theorem 4.23 says that there is a value x where $g(x) = 0$, and so $f(x) = x$.

4. Rudin, Chapter 4 #18

   Solution We will show that for any $x \in \mathbb{R}$, rational or irrational,

$$\lim_{y \to x} f(y) = 0\,.$$

   From this it follows immediately that f is continuous for irrational x, where $f(x) = 0$, and discontinuous if x is rational, since then $f(x) \neq 0$. Furthermore, the discontinuity is of the first kind, since $f(x+) = 0 = f(x-)$; that is, those limits are defined but not equal to $f(x)$.

   To calculate $\lim f(y)$, fix an $\varepsilon > 0$ and a value $N > 1/\varepsilon$. Let

$$A = \{a \mid a = \frac{m}{n}, n < N \quad \text{and} \quad |a - x| < 1\}\,.$$

   (The choice of 1 is arbitrary but convenient.) The set A is finite, since there are only a finite number of rationals in a given interval with denominators less than a fixed number. Hence the minimum value

$$d = \min_{a \in A, a \neq x} \{|a - x|\}$$

   is positive. Hence, for all $y \neq x$ with $|y - x| < d$ we have $|f(y)| < \varepsilon$, since either $y \notin \mathbb{Q}$ and $f(y) = 0$ or $y \in \mathbb{Q}$ and $y = p/q$ with $q > N$, so $f(y) = 1/q < \varepsilon$.

   Thus, $\lim_{y \to x} f(y) = 0$ as claimed.

5. Rudin, Chapter 4 #23

Solution In this question it is interesting to note that a convex function defined on a closed interval $[a, b]$ may be discontinuous at the end points of the interval. Since we are dealing with an open interval here, we know that for any $z \in (a, b)$ there exist $a < y < z < x < b$.

We begin by writing $z = \lambda x + (1 - \lambda)y$ for such values x, y, z and solving for λ:

$$\lambda = \frac{z - y}{x - y}.$$

The fundamental relationship, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, then becomes

$$f(z) \leq \frac{z - y}{x - y} f(x) + \left(1 - \frac{z - y}{x - y}\right) f(y),$$

and this can be rearranged in several ways, after multiplying through by $x - y$ (which is positive) to obtain

$$(z - y)f(x) + (x - z)f(y) - (x - y)f(z) \geq 0, \tag{0.1}$$

which is the same as

$$(z - y)\big(f(x) - f(y)\big) - (x - y)\big(f(z) - f(y)\big) \geq 0.$$

Dividing by $z - y$ and $x - y$ (which are positive) gives

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(z) - f(y)}{z - y}. \tag{0.2}$$

Another rearrangement in (0.1) gives

$$-(x - z)\big(f(x) - f(y)\big) + (x - y)\big(f(x) - f(z)\big) \geq 0,$$

and dividing by the positive factors and combining with (0.2) gives

$$\frac{f(x) - f(z)}{x - z} \geq \frac{f(x) - f(y)}{x - y} \geq \frac{f(z) - f(y)}{z - y}. \tag{0.3}$$

This is the same (with different letters for the variables) as the inequality that you are asked to prove in the problem.

We use (0.3) to prove continuity by thinking of x and y as fixed and taking limits as z increases to x or decreases to y. The term in the middle of (0.3) is independent of z and is some fixed, finite number, whether f is continuous or not. Let it be denoted by L. Then we have, using the inequality on the right in (0.3),

$$f(z) - f(y) \leq L(z - y),$$

and if we take the limit as z decreases to y we get

$$f(y+) - f(y) \leq 0 \quad \text{or} \quad f(y+) \leq f(y)\,,$$

since $L(z-y) \to 0$. Similarly, using the left inequality and letting z increase to x we get

$$f(x) - f(x-) \geq 0 \quad \text{or} \quad f(x-) \leq f(x)\,.$$

Since x and y are any points, we see that for any point in (a, b) we have

$$f(z-) \leq f(z) \quad \text{and} \quad f(z+) \leq f(z)\,.$$

On the other hand, if we return to (0.1) and take any $y < z < x$ with $x - z = z - y$ we get

$$2f(z) \leq f(x) + f(y)\,,$$

and so, letting x and y approach z at the same rate, we have

$$2f(z) \leq f(z+) + f(z-)\,.$$

If we suppose that $f(z+) \neq f(z)$ then we must have $f(z+) < f(z)$ by the earlier arguments, and similarly with $f(z-)$, and so we get

$$2f(z) \leq f(z+) + f(z-) < 2f(z)\,,$$

if either equality fails. Since this is impossible, we must have equality in both limits, and hence f is continuous at z, which is any point in (a, b).

To show that if g is increasing and convex then $h = g \circ f$ is also convex, we calculate

$$h(\lambda x + (1-\lambda)y) = g\big(f(\lambda x + (1-\lambda)y)\big) \leq g(\lambda f(x) + (1-\lambda)f(y)\big)\,,$$

which follows from the inequality for f and the fact that g is increasing. Now the conclusion follows from the convexity of g:

$$g(\lambda f(x) + (1-\lambda)f(y)\big) \leq \lambda g\big(f(x)\big) + (1-\lambda)g\big(f(y)\big) = \lambda h(x) + (1-\lambda)h(y)$$

as required.