X.23

(a) Let $c > b$ be given. We will show that $B \le c$. Since $\frac{b+c}{2} > \limsup x_n$, there exists N such that

$$x_n \le \frac{b+c}{2} \quad \text{whenever } n \ge N.$$

For any $m \ge \frac{2(x_1 + \dots + x_N)}{c-b}$, we have

$$\begin{aligned} s_m &= \frac{x_1 + \dots + x_N}{m} + \frac{x_{N+1} + \dots + x_m}{m} \\ &\le \frac{c-b}{2} + \frac{1}{m}(m-N)\frac{b+c}{2} \\ &\le \frac{c-b}{2} + \frac{b+c}{2} \\ &= c. \end{aligned}$$

Thus $B = \limsup s_n \le c$.

(b) Suppose $x_n \to L$. Then $a = b = L$.

Since $a \le A \le B \le b$ in general,

we have $A = B = L$. Thus $s_n \to L$.