Name:____________________________________

MATH 5201 Intro to Real Analysis I
AUTUMN 2015 FINAL EXAM
10:00am - 11:45am, December 15, 2015

Please choose 7 out of 8 problems from below. Please circle the problem numbers you have chosen on the front page, otherwise only the first 7 attempted problems will be graded. Each problem counts for 25 points.

No books, notes, or calculators are allowed.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
Total	

(1) Let $f : [-1, 1] \to \mathbb{R}$ be given.
  (a) (15 points) Suppose f is differentiable on $[-1, 1]$, and f' is continuous on $[-1, 1]$. Show that there exists $M \geq 0$ such that

$$|f(x) - f(y)| \leq M|x - y| \quad \text{for any } x, y \in [-1, 1].$$

  (b) (10 points) Prove or give a counter example to the following statement: Suppose $|f(x) - f(y)| \leq |x - y|$ holds for any $x, y \in [-1, 1]$, then f is differentiable in $[-1, 1]$.

(2) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{1}{(2n)!} z^n$.

(3) If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{x \in E} d(x, z).$$

Show that ρ_E is uniformly continuous on X.
[Hint: Show that $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$.]

(4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing, differentiable function, and that its inverse f^{-1} is also differentiable on the range of f. If $f(1) = 2$ and $(f^{-1})'(2) = 3$, find $f'(1)$.

(5) Suppose

$$f(x) = x \quad \text{and} \quad \alpha(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1; \\ x^2 & \text{for } 1 \leq x \leq 2. \end{cases}$$

(i) (5 points) For each $\epsilon > 0$, find $\int_0^{1-\epsilon} f\, d\alpha$.

(ii) (10 points) Find $\int_1^2 f\, d\alpha$.

(iii) (5 points) For each $\epsilon > 0$, show that $1 - \epsilon \leq \int_{1-\epsilon}^1 f\, d\alpha \leq 1$.

(iv) (5 points) Find $\int_0^2 f\, d\alpha$.

(6) Let f be a real-valued function defined on $[a, b]$ and α is an increasing function defined on $[a, b]$. Suppose $-f^5 \in \mathcal{R}(\alpha)$ on $[a, b]$. Does it follow that $f \in \mathcal{R}(\alpha)$?

(7) The radius of convergence of $f(x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ and $g(x) = \sum_{n=0}^{\infty} x^n$ are both 1.

Show that

$$f'(x) = g(x) \quad \text{for all } x \in (-1, 1).$$

[Hint: Let $f_N(x) = \sum_{n=1}^{N+1} \frac{1}{n} x^n$ and $g_N(x) = \sum_{n=0}^{N} x^n$. Then $(f_N)'(x) = g_N(x)$ for all x and for all $N \geq 1$. The assertion follows upon showing, for each $\delta > 0$, "appropriate" convergences of $f_N \to f$ and $g_N \to g$ on $[-1 + \delta, 1 - \delta]$. (15 points will be awarded if you can state the convergences correctly, 5 points for the verification of each convergence.)]

(8) Suppose $\{f_n\}$ is a sequence of uniformly bounded, complex-valued, Riemann-integrable functions on $[0,1]$. Put

$$F_n(x) = \int_0^x f_n(t)\,dt.$$

Show that there is a subsequence $\{F_{n_k}\}$ that converges uniformly on $[0,1]$.