Theorem. *Let I and J be sets, let Y be a metric space, and let $w\colon I \times J \to Y$. Let $\mathscr{A}$ and $\mathscr{B}$ be filter bases on I and J respectively. Suppose that*

$$w(i,j) \to u(i) \text{ in } Y \text{ as } j \to \mathscr{B}, \text{ for each } i \in I.$$

Suppose also that

$$w(i,j) \to v(j) \text{ in } Y \text{ uniformly for } j \in J \text{ as } i \to \mathscr{A}.$$

Then:

(a) *If either of the iterated limits*

$$\lim_{i \to \mathscr{A}} u(i) = \lim_{i \to \mathscr{A}} \lim_{j \to \mathscr{B}} w(i,j) \qquad \text{and} \qquad \lim_{j \to \mathscr{B}} v(j) = \lim_{j \to \mathscr{B}} \lim_{i \to \mathscr{A}} w(i,j)$$

exists in Y, then so does the other, and they are equal.

(b) *If the metric space Y is complete, then both of the limits in part* (a) *do exist* (*and so by part* (a), *they are equal*).

Proof. First suppose $u(i) \to y \in Y$ as $i \to \mathscr{A}$. Now for each $i \in I$ and each $j \in J$, we have

$$d(v(j), y) \le d(v(j), w(i,j)) + d(w(i,j), u(i)) + d(u(i), y). \tag{1}$$

Let $\varepsilon > 0$. Since $w(i,j) \to v(j)$ uniformly for $j \in J$ as $i \to \mathscr{A}$, there exists $A_1 \in \mathscr{A}$ such that for each $i \in A_1$, for each $j \in J$, $d(w(i,j), v(j)) \le \varepsilon/3$. Since $u(i) \to y \in Y$ as $i \to \mathscr{A}$, there exists $A_2 \in \mathscr{A}$ such that for each $i \in A_2$, $d(u(i), y) \le \varepsilon/3$. Since $\mathscr{A}$ is a filter base on I, there exists $A \in \mathscr{A}$ such that $A \subseteq A_1 \cap A_2$. Also since $\mathscr{A}$ is a filter base, $A \ne \emptyset$. Fix $i \in A$. Since $w(i,j) \to u(i)$ as $j \to \mathscr{B}$, there exists $B \in \mathscr{B}$ such that for each $j \in B$, $d(w(i,j), u(i)) \le \varepsilon/3$. Let $j \in B$. By (1), since $i \in A_1$, $j \in B$, and $i \in A_2$, we have $d(v(j), y) \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus for each $\varepsilon > 0$, there exists $B \in \mathscr{B}$ such that for each $j \in B$, $d(v(j), y) \le \varepsilon$. Therefore $v(j) \to y$ as $j \to \mathscr{B}$. This proves half of (a).

Conversely, suppose $v(j) \to y \in Y$ as $j \to \mathscr{B}$. Now for each $i \in I$ and each $j \in J$, we have

$$d(u(i), y) \le d(u(i), w(i,j)) + d(w(i,j), v(j)) + d(v(j), y). \tag{2}$$

Let $\varepsilon > 0$. Since $w(i,j) \to v(j)$ uniformly for $j \in J$ as $i \to \mathscr{A}$, there exists $A \in \mathscr{A}$ such that for each $i \in A$, for each $j \in J$, $d(w(i,j), v(j)) \le \varepsilon/3$. Let $i \in A$. Since $w(i,j) \to u(i)$ as $j \to \mathscr{B}$, there exists $B_1 \in \mathscr{B}$ such that for each $j \in B_1$, $d(w(i,j), u(i)) \le \varepsilon/3$. Since $v(j) \to y \in Y$ as $j \to \mathscr{B}$, there exists $B_2 \in \mathscr{B}$ such that for each $j \in B_2$, $d(v(j), y) \le \varepsilon/3$. Since $\mathscr{B}$ is a filter base, there exists $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2$. Also since $\mathscr{B}$ is a filter base, $B \ne \emptyset$. Fix $j \in B$. By (2), Since $j \in B_1$, $i \in A$, and $j \in B_2$, we have $d(u(i), y) \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus for each $\varepsilon > 0$, there exists $A \in \mathscr{A}$ such that for each $i \in A$, $d(u(i), y) \le \varepsilon$. Therefore $u(i) \to y$ as $i \to \mathscr{A}$. This proves the other half of (a).

Now suppose Y is complete. In view of (a), to prove that both of the limits in (a) exist in Y, it suffices to prove that one of them does. We claim that $u(i)$ converges in Y as $i \to \mathscr{A}$. Since Y is complete, it suffices to prove that $u(i)$ is Cauchy in Y as i runs along $\mathscr{A}$. Now for all $i, i' \in I$ and for each $j \in J$,

$$d(u(i), u(i')) \le d(u(i), w(i,j)) + d(w(i,j), v(j)) + d(v(j), w(i',j)) + d(w(i',j), u(i')). \tag{3}$$

Let $\varepsilon > 0$. Since $w(i,j) \to v(j)$ uniformly for $j \in J$ as $i \to \mathscr{A}$, there exists $A \in \mathscr{A}$ such that for each $i \in A$, for each $j \in J$, $d(w(i,j), v(j)) \le \varepsilon/4$. Let $i, i' \in A$. Since $w(i,j) \to u(i)$ as $j \to \mathscr{B}$, there exists $B_1 \in \mathscr{B}$ such that for each $j \in B_1$, $d(u(i), w(i,j)) \le \varepsilon/4$. Similarly, since $w(i',j) \to u(i')$ as $j \to \mathscr{J}$, there exists $B_2 \in \mathscr{B}$ such that for each $j \in B_2$, $d(w(i',j), u(i')) \le \varepsilon/4$. Since $\mathscr{B}$ is a filter base, there exists $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2$. Also since $\mathscr{B}$ is a filter base, $B \ne \emptyset$. Fix $j \in B$. By (3), since $j \in B_1$, $i, i' \in A$, and $j \in B_2$, we have $d(u(i), u(i')) \le \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon$. Thus for each $\varepsilon > 0$, there exists $A \in \mathscr{A}$ such that for all $i, i' \in A$, $d(u(i), u(i')) \le \varepsilon$. Therefore $u(i)$ is Cauchy in Y as i runs along $\mathscr{A}$. This proves (b). ■

Example. Let X be a topological space, let Y be a metric space, let $f, f_1, f_2, \ldots$ be functions from X to Y, and let $a \in X$.

(c) If f_n is continuous at a for each $n \in \mathbf{N}$ and if $f_n \to f$ uniformly on X as $n \to \infty$, then f is continuous at a.

(d) If $f_n \to f$ pointwise on X as $n \to \infty$ and if $\{\, f_n : n \in \mathbf{N} \,\}$ is equicontinuous at a, then f is continuous at a.

Proof. First let's prove part (a). Suppose each f_n is continuous at a and $f_n \to f$ uniformly on X as $n \to \infty$. Let $\mathscr{B}$ be the filter base of nhds of a in X. Since each f_n is continous at a, $f_n(x) \to f_n(a)$ as $x \to \mathscr{B}$, for each $n \in \mathbf{N}$. Now $\lim_{n\to\infty} \lim_{x\to\mathscr{B}} f_n(x) = \lim_{n\to\infty} f_n(a)$ exists in Y and is equal to $f(a)$. Therefore by the theorem, $\lim_{x\to\mathscr{B}} \lim_{n\to\infty} f_n(x)$ exists in Y and is equal to $f(a)$ too. In other words, $\lim_{x\to\mathscr{B}} f(x) = f(a)$, so f is continuous at a.

Now let's prove part (b). So suppose instead that $f_n(x) \to f(x)$ as $n \to \infty$, for each $x \in X$, and that $\{ f_n : n \in \mathbf{N} \}$ is equicontinuous at a. Again, let $\mathscr{B}$ be the filter base of nhds of a in X. To say that $\{ f_n : n \in \mathbf{N} \}$ is equicontinuous at a means that for each $\varepsilon > 0$, there exists a nhd B of a in X such that for each $x \in B$, for each $n \in \mathbf{N}$, $d(f_n(x), f_n(a)) < \varepsilon$. Thus $f_n(x) \to f_n(a)$ uniformly for $n \in \mathbf{N}$ as $x \to \mathscr{B}$. Now $\lim_{n\to\infty} \lim_{x\to\mathscr{B}} f_n(x) = \lim_{n\to\infty} f_n(a)$ exists in Y and is equal to $f(a)$. Therefore by the theorem, $\lim_{x\to\mathscr{B}} \lim_{n\to\infty} f_n(x)$ exists in Y and is equal to $f(a)$ too. In other words, $\lim_{x\to\mathscr{B}} f(x) = f(a)$, so f is continuous at a. ■

Example. Let Y be a metric space and let (f_n) be a sequence of functions from $\mathbf{R}$ to Y which converges uniformly on $\mathbf{R}$ to a function $f\colon \mathbf{R} \to Y$.

- **(e)** Let $a \in \mathbf{R}$. Suppose that each f_n is right-continuous at a. Then f is right-continuous at a. Similarly for left-continuous.
- **(f)** Let $a \in [-\infty, \infty)$. Suppose that for each $n \in \mathbf{N}$, the right limit $f_n(a+) = \lim_{x\to a+} f_n(x)$ exists in Y. If either of the limits

$$f(a+) = \lim_{x\to a+} f(x) \qquad \text{and} \qquad \lim_{n\to\infty} f_n(a+)$$

exists in Y, then both of them do and they are equal. If in addition, Y is complete, then both of these limits do exist in Y (and so they are equal). Similarly for limits as $x \to b-$, where $b \in (-\infty, \infty]$.

Example. Let (ξ_k) be a sequence of distinct points in $\mathbf{R}$. Any such sequence will do, but this example will be the most interesting if we choose this sequence so that its range $B = \{ \xi_k : k \in \mathbf{N} \}$ is dense in $\mathbf{R}$. For instance, (ξ_k) might be an enumeration of the rationals. Let $A = \mathbf{R} \setminus B$. For each $k \in \mathbf{N}$, let $u_k, v_k, w_k \in \mathbf{R}$, define $g_k\colon \mathbf{R} \to \mathbf{R}$ by

$$g_k(x) = \begin{cases} u_k & \text{if } x < \xi_k, \\ v_k & \text{if } x = \xi_k, \\ w_k & \text{if } x > \xi_k, \end{cases}$$

and let $M_k = \max\{ |u_k|, |v_k|, |w_k| \}$, so that for each $x \in \mathbf{R}$, we have $|g_k(x)| \le M_k$. Suppose that $\sum_{k=1}^{\infty} M_k < \infty$. Then the series $f = \sum_{k=1}^{\infty} g_k$ converges uniformly on $\mathbf{R}$. In other words, the sequence of its partial sums $f_n = \sum_{k=1}^{n} g_k$ converges uniformly to f on $\mathbf{R}$. It clearly follows from the previous example that f is continuous at each point of A, that f has left and right limits at each point of B, and since the points ξ_k are distinct,

$$f(\xi_k+) - f(\xi_k) = \sum_{\ell=1}^{\infty} (g_\ell(\xi_k+) - g_\ell(\xi_k)) = g_k(\xi_k+) - g_k(\xi_k) = w_k - v_k$$

and

$$f(\xi_k) - f(\xi_k-) = \sum_{\ell=1}^{\infty} (g_\ell(\xi_k) - g_\ell(\xi_k-)) = g_k(\xi_k) - g_k(\xi_k-) = v_k - u_k.$$

This construction thus provides us with a function f which has fairly arbitrary prescribed jumps at each point of a given countable set $B \subseteq \mathbf{R}$ but which is continuous at each point of $A = \mathbf{R} \setminus B$.