Example. To show that

$$\lim_{h\to 0} \frac{\sin h}{h} = 1, \tag{1}$$

you should not use l'Hospital's rule. In most calculus textbooks, the limit (1) is used in the proof that

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{ds}\cos x = -\sin x. \tag{2}$$

If that is your way of proving (2), and if you use l'Hospital's rule to prove (1), then you are guilty of circular reasoning. Most calculus textbooks prove (1) by using a geometrical argument to show that

$$\cos h < \frac{\sin h}{h} < 1 \tag{3}$$

for $h \in [-\pi/2, 0) \cup (0, \pi/2]$, and then by using the squeeze theorem to deduce (1) from (3).

In analysis textbooks, $\sin x$ and $\cos x$ are often defined by their power series and (2) is justified by termwise differentiation of these series. Even if this is your way of proving (2), you should not use l'Hospital's rule to prove (1). Instead, just notice that since $\sin 0 = 0$, the limit in (1) is precisely

$$\lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \sin' 0 = \cos 0 = 1.$$

Example. Let $a \in (0, \infty)$. It would be silly to use l'Hospital's rule to evaluate the limit

$$\lim_{p\to 0} \frac{a^p - 1}{p}.$$

In fact, since $1 = a^0$, this limit is just the derivative of the function $p \mapsto a^p$ at $p = 0$. We have

$$\frac{d}{dp}a^p = \frac{d}{dp}\exp(p\log a) = \exp(p\log a)\cdot \log a = a^p \log a,$$

so $\lim\limits_{p\to 0} \dfrac{a^p - 1}{p} = a^0 \log a = \log a$.

Example. It would be silly to use l'Hospital's rule to evaluate the limit

$$\lim_{h\to 0} \frac{1}{h}\log(1+h).$$

In fact, since $\log 1 = 0$, this limit is just $\lim\limits_{h\to 0} \dfrac{\log(1+h) - \log 1}{h} = \log' 1 = \dfrac{1}{1} = 1$.

Remark. More generally, if g is defined in a neighborhood of a real number a and if g is differentiable at a and $g(a) = 0$, then

$$\lim_{h\to 0} \frac{g(a+h)}{h}$$

is just $g'(a)$. This holds even if g is not differentiable anywhere but at a, in which case l'Hospital's rule does not even apply.

Example. To show that

$$\lim_{x\to\infty} \frac{e^x}{x} = \infty, \tag{4}$$

you could use l'Hospital's rule. However, this would not help you to determine how large x should be to ensure that e^x/x exceeds some given value. But since the function $x \mapsto e^x$ is convex on $\mathbf{R}$, its graph lies above each of its tangent lines. Considering the line that is tangent to this graph at $x = 1$, we see that

$$e^x \geq ex.$$

Let $0 < a < 1$ and let $b = 1 - a$. Then

$$\frac{e^x}{x} = e^{ax}\frac{e^{bx}}{x} \geq e^{ax}\frac{ebx}{x} = eb \cdot e^{ax}. \tag{5}$$

Obviously (4) follows from (5). Furthermore, if $M \in (0, \infty)$ and if

$$x \geq \frac{1}{a}\log\Big(\frac{M}{eb}\Big),$$

then $e^x/x \geq M$. In a similar manner, for any $k, p \in (0, \infty)$, you can obtain lower bounds on e^{kx}/x^p and show that $e^{kx}/x^p \to \infty$ as $x \to \infty$.

Example. To show that
$$\lim_{x\to\infty} \frac{\log x}{x} = 0, \tag{6}$$
you could use l'Hospital's rule. However, this would not help you to determine how large x should be to ensure that $(\log x)/x$ is less than some given positive value. But since the function $x \mapsto \log x$ is concave on $(0, \infty)$, its graph lies below each of its tangent lines. Considering the line that is tangent to this graph at $x = e$, we see that
$$\log x \leq \frac{x}{e}.$$
Let $0 < a < 1$ and let $b = 1 - a$. Then
$$\frac{\log x}{x} = \frac{b^{-1} \log x^b}{x^a x^b} \leq \frac{x^b}{ebx^a x^b} = \frac{1}{ebx^a}. \tag{7}$$
Obviously (6) follows from (7). Furthermore, if $\varepsilon \in (0, \infty)$ and if
$$x \geq \left(\frac{1}{eb\varepsilon}\right)^{1/a},$$
then $(\log x)/x \leq \varepsilon$. In a similar manner, for each $p \in (0, \infty)$, you can obtain upper bounds on $(\log x)/x^p$ and show that $(\log x)/x^p \to 0$ as $x \to \infty$. Since $\log(1/x) = -\log x$, you can also use these bounds to show that for each $p \in (0, \infty)$, $x^p \log x \to 0$ as $x \to 0+$.

A Good Use of l'Hospital's Rule. Let $a \in \mathbf{R}$. Let f be a real-valued function that is defined and $n-1$ times differentiable in a neighbourhood of a and suppose $f^{(n)}(a)$, the n-th derivative of f at a, exists. Under these assumptions, we are going to prove Taylor's formula with the "little oh" form of the remainder. Let P be the n-th order Taylor polynomial for f at a. In other words, define $P\colon \mathbf{R} \to \mathbf{R}$ by
$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$
Let R be the n-th order Taylor remainder for f at a. In other words, let $R = f - P$, so that $f = P + R$. Then
$$R(x) = o\big((x-a)^n\big) \qquad \text{as } x \to a.$$
In other words,
$$\frac{R(x)}{(x-a)^n} \to 0 \qquad \text{as } x \to a.$$
The proof is easy: Just apply l'Hospital's rule $n-1$ times and then apply the definition of $f^{(n)}(a)$. (Warning: It would not work just to apply l'Hospital's rule n times, because $f^{(n)}(x)$ need not be defined when $x \neq a$ and even if $f^{(n)}(x)$ is defined for all x in some nhd of a, $f^{(n)}(x)$ need not tend to $f^{(n)}(a)$ as $x \to a$.)

Remark. It is not much of an exaggeration to say that the version of Taylor's formula just discussed is the only application of l'Hospital's rule that is not silly. Anytime you find yourself using l'Hospital's rule to evaluate a concrete limit, you should stop and ask yourself whether there is any good reason for you to appeal to l'Hospital's rule.