Quiz 3 MATH 5201, AUTUMN 2016 NAME: Key

(circle one): Auditing Registered

Problem 1.

(a) Let $\{p_n\}$ be a sequence in a metric space (X, d). Prove that if $\{p_n\}$ is convergent, then $\{p_n\}$ is Cauchy. (Justify each step).

Let p denote the limit pt.

Let $\varepsilon > 0$. Choose N s.t. if $n \geq N$, then $|p - p_n| < \frac{\varepsilon}{2}$. Now, $m, n \geq N$ implies

$$|p_m - p_n| \leq |p - p_m| + |p - p_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

$\therefore$ {Convergent} $\longrightarrow$ {Cauchy}.

(b) Give an example to show that the converse need not be true. (Explain why).

Consider the Cauchy sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ in $(0,1)$ with the Euclidean metric.

This sequence is Cauchy. Indeed, let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $2 < \varepsilon \cdot N$ (Archimedean principle). Now, $\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| < \varepsilon$ provided $n, m \geq N$.

This sequence d.n. converge in $(0,1)$ b/c $0 \notin (0,1)$.