Quiz 7 Prove that if f is a continuous bijection from $\mathbb{R}$ to $\mathbb{R}$, then $f(\mathbb{Q})$ dense.

example: $f(x) = \tan^{-1} x$, $x \in (-1,1)$

(note: One only needs that f is a continuous surjective function, but we allow easier hypotheses for more variety of proof)

pf// $\boxed{\text{proof 1}}$ $\mathbb{R} = f(\mathbb{R}) = f(\overline{\mathbb{Q}}) \subseteq \overline{f(\mathbb{Q})}$

Uses f onto & f cont.

(H.W. Rudin Chapt. 4 #2)

A set is dense in $\mathbb{R}$ if its closure equals $\mathbb{R}$ ✓

We show that $\exists\ x \in \mathbb{Q}$ s.t. $y_1 < x < y_2$

$\boxed{\text{Proof 2}}$ Let $y_1 < y_2$. f^{-1} bijection and so $f^{-1}(y_1) \neq f^{-1}(y_2)$. Let p/q inbetween $f^{-1}(y_1)$ & $f^{-1}(y_2)$, $\frac{p}{q} \in \mathbb{Q}$.

applied to f^{-1}

By the I.V.T., $\exists\ x \in (y_1, y_2)$ s.t. $f^{-1}(x) = \frac{p}{q}$. Now, $x = f(\frac{p}{q}) \in f(\mathbb{Q})$.

$\boxed{\text{Proof 3}}$ Use that the inverse image of open is open.

Let $x \in \mathbb{R}$. Then $f(x) \in \mathbb{R}$. Let $\mathcal{O}$ open about $f(x)$.

We show $\exists\ \frac{p}{q} \in \mathbb{Q}$ s.t. $f(\frac{p}{q}) \in \mathcal{O}$.

f cont. bijection $\rightarrow$ f^{-1} is defined & continuous. $\Rightarrow$

$f^{-1}(\mathcal{O})$ is open & $f^{-1}(\mathcal{O}) \neq \emptyset$. Let $\frac{p}{q} \in f^{-1}(\mathcal{O})$.

Then $f(\frac{p}{q}) \in \mathcal{O} \cap f(\mathbb{Q})$ ✓

uses f onto & f cont. → $\boxed{\text{proof 4}}$ Let $y \in \mathbb{R}$. Write $y = f(x)$.

Let $\varepsilon > 0$. Then $\exists$ $\delta > 0$ s.t. if $d(x, p) < \delta$, then $d(f(x), f(p)) < \varepsilon$. Choose $p \in \mathbb{Q}$ s.t. $d(x, p) < \delta$. Then $d(f(x), f(p)) < \varepsilon$. ✓