A Review of Riemann Integration

This review is intended for students who are beginning the study of Lebesgue integration. It is expected that such students will already have studied Riemann integration. They should feel free to skim this review and to refer back to it later as needed.

Let $a, b \in \mathbf{R}$ with $a < b$. If Y is a set and $\varphi\colon [a,b] \to Y$, then to say that φ *is a step function* means that $[a,b]$ can be partitioned into finitely many intervals on each of which φ is constant. If $\varphi_1, \ldots, \varphi_n$ are finitely many step functions, then, by considering suitable intersections of intervals, it is easy to see that $[a,b]$ can be partitioned into finitely many invervals on each of which all of $\varphi_1, \ldots, \varphi_n$ are constant. It follows that any way of combining finitely many step functions yields another step function. For instance, if $\varphi_1, \varphi_2\colon [a,b] \to \mathbf{R}$ are step functions, then so are $\varphi_1 + \varphi_2$ and $\varphi_1 \vee \varphi_2 = \max\{\,\varphi_1, \varphi_2\,\}$. Let us write Φ for the set of step functions $\varphi\colon [a,b] \to \mathbf{R}$. For each $\varphi \in \Phi$, we have

$$\varphi = \sum_{k=1}^{n} c_k 1_{I_k}, \tag{1}$$

where $n \in \mathbf{N}$, $c_1, \ldots, c_n \in \mathbf{R}$, and $I_1, \ldots, I_n$ are disjoint subintervals of $[a,b]$, 1_{I_k} denotes the indicator of I_k, or in other words, the function that takes the value 1 on I_k and 0 elsewhere, and we define the integral of φ (over $[a,b]$) to be

$$I(\varphi) = \sum_{k=1}^{n} c_k |I_k|,$$

where $|I_k|$ denotes the length of the interval I_k. Since a given step function $\varphi \in \Phi$ can be expressed in the form (1) in many different ways, one should verify that $I(\varphi)$ is well-defined. I'll leave it to you to verify that $I(\varphi)$ is indeed well-defined. (However, before long we shall give a careful proof of a more general result). I'll also leave it to you to verify that I is a linear map from Φ into $\mathbf{R}$. In other words, for all $\varphi, \psi \in \Phi$ and for all $c \in \mathbf{R}$, we have $I(\varphi + \psi) = I(\varphi) + I(\psi)$ and $I(c\varphi) = cI(\varphi)$. (Again, before long we shall give a careful proof of a more general result.) Clearly for each $\varphi \in \Phi$, if $\varphi \geq 0$, then $I(\varphi) \geq 0$. Hence for all $\varphi, \psi \in \Phi$, if $\varphi \leq \psi$, then $I(\varphi) \leq I(\psi)$, because $I(\psi) - I(\varphi) = I(\psi - \varphi) \geq 0$.

Definitions. Let $f\colon [a,b] \to \mathbf{R}$. Then the *lower Riemann integral of f (over $[a,b]$)* is

$$R_*(f) = \sup \Phi_*(f),$$

where $\Phi_*(f) = \{\, I(\varphi) : \varphi \in \Phi \text{ and } \varphi \leq f \,\}$, and the *upper Riemann integral of f (over $[a,b]$)* is

$$R^*(f) = \inf \Phi^*(f),$$

where $\Phi^*(f) = \{\, I(\varphi) : \varphi \in \Phi \text{ and } f \leq \varphi \,\}$.

Remark. Let $f \in \Phi$. Then for all $\varphi, \psi \in \Phi$ with $\varphi \leq f \leq \Psi$, we have $I(\varphi) \leq I(f) \leq I(\psi)$. It follows that $R_*(f) = I(f) = R^*(f)$.

Remarks. Let $f\colon [a,b] \to \mathbf{R}$. Notice that if f is bounded below, say by $M_1 \in \mathbf{R}$, then $R_*(f) \geq M_1(b-a) > -\infty$. However, if f is not bounded below, then $\Phi_*(f)$ is empty, so $R_*(f) = \sup \varnothing = -\infty$. Similarly, if f is bounded above, say by $M_2 \in \mathbf{R}$, then $R^*(f) \leq M_2(b-a) < \infty$, but if f is not bounded above, then $\Phi^*(f)$ is empty, so $R^*(f) = \inf \varnothing = \infty$.

Exercise 1. Let $f\colon [a,b] \to \mathbf{R}$. Prove that $R_*(f) \leq R^*(f)$.

Exercise 2. Let $f, g\colon [a,b] \to \mathbf{R}$ be bounded. Prove that

$$R^*(f+g) \leq R^*(f) + R^*(g) \qquad \text{and} \qquad R_*(f) + R_*(g) \leq R_*(f+g). \tag{2}$$

Exercise 3. Let $f\colon [a,b] \to \mathbf{R}$ be bounded and let $c \in [0,\infty)$. Prove that

$$R^*(cf) = cR^*(f), \qquad R_*(cf) = cR_*(f), \qquad R^*(-cf) = -cR_*(f) \qquad \text{and} \qquad R_*(-cf) = -cR^*(f). \tag{3}$$

Definitions. Let $f\colon [a,b] \to \mathbf{R}$. To say that f *is Riemann-integrable (over $[a,b]$)* means that

$$-\infty < R_*(f) = R^*(f) < \infty,$$

and in this case, the *Riemann integral of f* is defined to be

$$\int_a^b f(x)\,dx = R_*(f) = R^*(f).$$

Warning. Let $f\colon [a,b] \to \mathbf{R}$ be Riemann-integrable. Then in particular, $R_*(f) > -\infty$ and $R^*(f) < \infty$, so f must be a bounded function. We are not considering improper integrals here.

A Review of Riemann Integration

Remark. Let $f, g\colon [a,b] \to \mathbf{R}$ be Riemann-integrable. From (2), it follows easily that $f+g$ is Riemann-integrable and

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \tag{4}$$

From (3), it follows easily that for each $c \in \mathbf{R}$, cf is Riemann-integrable and

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx. \tag{5}$$

Remark. Let $h\colon [a,b] \to \mathbf{R}$ be Riemann-integrable and let $u \in (a,b)$. Then

$$\int_a^b h(x)\,dx = \int_a^u h(x)\,dx + \int_u^b h(x)\,dx.$$

This follows from (4) with $f = h1_{[a,u]}$ and $g = h1_{(u,b]}$. (Obviously $h = f+g$. It's easy to check that f and g are Riemann-integrable over $[a,b]$, that $\int_a^b f(x)\,dx = \int_a^u h(x)\,dx$, and that $\int_a^b g(x)\,dx = \int_u^b h(x)\,dx$.)

Regulated Functions. Let $f\colon [a,b] \to \mathbf{R}$. No doubt you are familiar with the fact that if f is continuous, then f is Riemann-integrable. We shall present a useful generalization of this fact. To say that f *is regulated* means that $\lim_{x\to c+} f(x)$ exists in $\mathbf{R}$ for each $c \in [a,b)$ and $\lim_{x\to d-} f(x)$ exists in $\mathbf{R}$ for each $d \in (a,b]$. For instance, any step function $\varphi\colon [a,b] \to \mathbf{R}$ is regulated. So is any piecewise continuous function, any monotone function, and any function of bounded variation. The next few exercises will lead you through a proof of the fact that if f is regulated, then f is Riemann-integrable.

Exercise 4. Let $E \subseteq [a,b]$. Suppose E is relatively right-open in $[a,b]$ and is also left-closed.[1] Prove that $E = [a,b]$.

Remark. The familiar facts that $[a,b]$ is both compact and connected are easy consequences of Exercise 4.

Exercise 5. Let $f\colon [a,b] \to \mathbf{R}$ be regulated. Let $\varepsilon > 0$. Prove that there exist step functions $\varphi, \psi\colon [a,b] \to \mathbf{R}$ such that for each $x \in [a,b]$, we have $\varphi(x) \le f(x) \le \psi(x)$ and $\psi(x) - \varphi(x) < \varepsilon$. (Hint: Let E be the set of all $c \in [a,b]$ such that such step functions exist on $[a,c]$ and apply Exercise 4.)

Remark. Conversely, if such step functions exist on $[a,b]$ for each $\varepsilon > 0$, then it is not hard to show that f is regulated. Thus we have a satisfying characterization of regulated functions in terms of their uniform approximability by step functions.

Exercise 6. Let $f\colon [a,b] \to \mathbf{R}$ be regulated. Prove that f is Riemann-integrable.

Exercise 7. Let $F\colon [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Let $f\colon [a,b] \to \mathbf{R}$ satisfy $f(x) = F'(x)$ for all $x \in (a,b)$.

(a) Prove that

$$R_*(f) \le F(b) - F(a) \le R^*(f).$$

(b) Deduce that if f is Riemann-integrable over $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

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[1] Let $E \subseteq \mathbf{R}$. To say that E *is right-open* means that for each $c \in E$, there exists $d > c$ such that $[c,d) \subseteq E$. The collection of right-open subsets of $\mathbf{R}$ is a topology on $\mathbf{R}$, called the right-topology on $\mathbf{R}$. It is worth mentioning that a function from $\mathbf{R}$ into a topological space Y is continuous with respect to the right-topology iff it is right-continuous in the usual sense. The *left-topology on* $\mathbf{R}$ and the term *left-open* are defined analogously. To say that E *is left-closed* means that E is closed with respect to the left-topology on $\mathbf{R}$, or in other words, that $\mathbf{R} \setminus E$ is left-open. It is easy to see that E is left-closed iff for each $c \in \mathbf{R} \setminus E$, there exists $d < c$ such that $E \cap (d,c]$ is empty. Each subset of $\mathbf{R}$ that is open with respect to the usual topology on $\mathbf{R}$ is both right-open and left-open. Each subset of $\mathbf{R}$ that is closed with respect to the usual topology on $\mathbf{R}$ is both right-closed and left-closed. To say that E *is right-open in* $[a,b]$ means that $E \subseteq [a,b]$ and E is open with respect to the topology that $[a,b]$ inherits from $\mathbf{R}$ with its right-topology. It is easy to see that E is right-open in $[a,b]$ iff for each $c \in E$, if $c < b$, then there exists $d \in (c,b]$ such that $[c,d) \subseteq E$. Since $[a,b]$ is closed with respect to the usual topology on $\mathbf{R}$, it is both right-closed and left-closed, so a subset of $[a,b]$ is right-closed (respectively, left-closed) in $[a,b]$ iff it is right-closed (respectively, left-closed) in $\mathbf{R}$.

A Review of Riemann Integration

Notation. Let X be a set and let $f\colon X \to [-\infty, \infty]$. Then f^+ and f^-, the positive and negative parts of f, are the functions on X defined by $f^+(x) = \max\{\, f(x), 0\,\}$ and $f^-(x) = \max\{\, -f(x), 0\,\}$. Note that $f^+ \ge 0$, $f^- \ge 0$, $f = f^+ - f^-$, and $|f| = f^+ + f^-$.

Exercise 8. Let $f\colon [a,b] \to \mathbf{R}$ be Riemann-integrable. Prove that f^+, f^-, and $|f|$ are Riemann-integrable.

Exercise 9. Let $f\colon [a,b] \to \mathbf{R}$ be Riemann-integrable. Prove the triangle inequality for integrals:

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Exercise 10. Let $f\colon [a,b] \to \mathbf{R}$ be Riemann-integrable and let $\varepsilon > 0$. Prove that there is a step function $\varphi\colon [a,b] \to \mathbf{R}$ such that $\int_a^b |f(x) - \varphi(x)|\,dx < \varepsilon$ and $|\varphi| \le |f|$.

Notation. For each bounded function $f\colon [a,b] \to \mathbf{R}$, let

$$\rho(f) = R^*(|f|).$$

Note that $\rho(f)$ is defined even if f is not Riemann-integrable. However, if f is Riemann-integrable, then $\rho(f) = \int_a^b |f(x)|\,dx$.

Remark. Let $\mathscr{B}$ be the vector space of bounded real-valued functions on $[a,b]$. It follows from (2) and (3) that ρ is a seminorm on $\mathscr{B}$. This means that for all $f, g \in \mathscr{B}$ and for all $c \in \mathbf{R}$, we have $0 \le \rho(f) < \infty$, $\rho(f+g) \le \rho(f) + \rho(g)$, and $\rho(cf) = |c|\rho(f)$. However, ρ is not a norm, because we can have $\rho(f) = 0$ without having $f = 0$. For instance, if $f\colon [a,b] \to \mathbf{R}$ and $f = 0$ at all but finitely many points, then $f \in \mathscr{B}$ and $\rho(f) = 0$. However, a function $f \in \mathscr{B}$ for which $\rho(f) = 0$ may be thought of as being the zero function for most practical purposes. As we learn more about integration theory, we'll develop a better understanding of this point.

Exercise 11. Let $f\colon [a,b] \to \mathbf{R}$ be bounded. Suppose (f_n) is a sequence of Riemann-integrable functions $f_n\colon [a,b] \to \mathbf{R}$ such that $\rho(f - f_n) \to 0$. Prove that f is Riemann-integrable and that

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Remark. Let (f_n) be a sequence of Riemann-integrable functions $f_n\colon [a,b] \to \mathbf{R}$ which converges uniformly to a function $f\colon [a,b] \to \mathbf{R}$. Then, as you should know, f is Riemann-integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$. Exercise 11 may be viewed as a generalization of this standard fact. For since $f_n \to f$ uniformly, there is a sequence (M_n) in $[0,\infty)$ such that $M_n \to 0$ and for each n, $|f - f_n| \le M_n$. Then for each n, $\rho(f - f_n) \le M_n(b-a) \to 0$, so by Exercise 11, f is Riemann-integrable and $\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$.

However, Exercise 11 is strictly more general. For instance, if $[a,b] = [0,1]$ and $|f(x) - f_n(x)| = x^{n-1}$, then $\rho(f - f_n) = \frac{1}{n} \to 0$ as $n \to \infty$ but (f_n) does not tend to f uniformly, because for each n, if M_n is a constant and $|f - f_n| \le M_n$, then $M_n \ge 1$.

Notation. Let $E \subseteq [a,b]$. Then *the outer content of* E is

$$\gamma^*(E) = R^*(1_E).$$

It's worth noticing that if $E = E_1 \cup \cdots \cup E_n$, then $\gamma^*(E) \le \gamma^*(E_1) + \cdots + \gamma^*(E_n)$ by (2), because $1_E \le 1_{E_1} + \cdots + 1_{E_n}$.

Exercise 12. Let $h\colon [a,b] \to [0,\infty)$ be bounded. Let $u \in (0,\infty)$. Write $\gamma^*(h \ge u)$ as an abbreviation for $\gamma^*(\{\, x : h(x) \ge u\,\})$. Then:

(a) $\gamma^*(h \ge u) \le \frac{1}{u} R^*(h)$. (Remark: This is a version of what is sometimes called *Markov's inequality.*)

(b) $R^*(h) \le M\gamma^*(h \ge u) + u \cdot (b-a)$, where $M = \sup h$.

(Hint: Each part has a one-line proof that is easier to remember than its statement.)

A Review of Riemann Integration

Exercise 13. Let $f\colon [a,b] \to \mathbf{R}$ be Riemann-integrable and let $g\colon \mathbf{R} \to \mathbf{R}$ be continuous. Prove that $g \circ f$ is Riemann-integrable. (Hint: Choose $M \in [0,\infty)$ such that $|f| \le M$. By Exercise 10, there is a sequence (φ_n) of step functions $\varphi_n\colon [a,b] \to \mathbf{R}$ such that $\rho(f - \varphi_n) \to 0$ as $n \to \infty$ and for each n, $|\varphi_n| \le M$. Note that g is uniformly continuous on $[-M, M]$. Use Exercise 12 to show that $\rho(g \circ f - g \circ \varphi_n) \to 0$ as $n \to \infty$. Then apply Exercise 11.)

Warning. Remember that we are considering only *properly* Riemann-integrable functions here. Exercise 13 does not apply if f is only improperly Riemann-integrable. For instance, consider the functions $f\colon (0,1] \to \mathbf{R}$ and $g\colon \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^{-1/2}$ and $g(y) = y^2$.

Example. Let $f\colon [a,b] \to \mathbf{R}$ be Riemann-integrable. Then f^2 is Riemann-integrable, because if we let $g(y) = y^2$ for all $y \in \mathbf{R}$, then g is continuous and $f^2 = g \circ f$. Now let $f_1, f_2\colon [a,b] \to \mathbf{R}$ be Riemann-integrable. Then their product $f_1 f_2 = \frac{1}{2}[(f_1 + f_2)^2 - f_1^2 - f_2^2]$ is Riemann-integrable.

Exercise 14. Let $h\colon [a,b] \to [0,\infty)$ be bounded.

- **(a)** Prove that $R^*(h) = 0$ if and only if for each $u \in (0,\infty)$, we have $\gamma^*(h \ge u) = 0$. (Hint: Use Exercise 12.)
- **(b)** Let $p \in (0,\infty)$. Prove that $R^*(h) = 0$ if and only if $R^*(h^p) = 0$. (Hint: Use part (a).)

Riemann Integration of $\mathbf{R}^d$-Valued Functions. Let $f\colon [a,b] \to \mathbf{R}^d$. Then there are unique functions $f_1, \ldots, f_d\colon [a,b] \to \mathbf{R}$ such that for each $x \in [a,b]$, $f(x) = (f_1(x), \ldots, f_d(x))$. The functions $f_1, \ldots, f_d$ are called the components of f. To say that f *is Riemann-integrable* (*over* $[a,b]$) means that $f_1, \ldots, f_d$ are all Riemann-integrable, and in this case, *the Riemann integral of* f (*over* $[a,b]$) is the point $y = (y_1, \ldots, y_d)$ in $\mathbf{R}^d$, where $y_k = \int_a^b f_k(x)\,dx$ for $k = 1, \ldots, d$. If f is Riemann-integrable, then f must be bounded, because $f_1, \ldots, f_d$ must be bounded. It's obvious that (4) and (5) hold for all Riemann-integrable functions $f, g\colon [a,b] \to \mathbf{R}^d$ and all $c \in \mathbf{R}$.

Notation. For each $y = (y_1, \ldots, y_d)$ in $\mathbf{R}^d$, the Euclidean norm of y is

$$|y| = (y_1^2 + \cdots + y_d^2)^{1/2}.$$

For all $y = (y_1, \ldots, y_d)$ and $z = (z_1, \ldots, z_d)$ in $\mathbf{R}^d$, the scalar product of y and z is

$$\langle y \,|\, z \rangle = y_1 z_1 + \cdots + y_d z_d.$$

Notice that for each $y = (y_1, \ldots, y_d)$ in $\mathbf{R}^d$, we have $|y| = \langle y \,|\, y \rangle^{1/2}$. Recall Schwarz's inequality: For all y and z in $\mathbf{R}^d$, $|\langle y \,|\, z \rangle| \le |y||z|$.

Remark. Let $f\colon [a,b] \to \mathbf{R}^d$ and let $f_1, \ldots, f_d$ be the components of f. Then $|f|$ is the function

$$x \mapsto |f(x)| = \bigl(f_1(x)^2 + \cdots + f_d(x)^2\bigr)^{1/2}$$

from $[a,b]$ into $[0,\infty)$. Suppose f is Riemann-integrable. Then $f_1, \ldots, f_d$ are Riemann-integrable, so by Exercise 13, $f_1^2, \ldots, f_d^2$ are all Riemann-integrable. Hence $f_1^2 + \cdots + f_d^2$ is Riemann-integrable. Thus, by Exercise 13 again, $|f| = (f_1^2 + \cdots + f_d^2)^{1/2}$ is Riemann-integrable.

Exercise 15. Let $f\colon [a,b] \to \mathbf{R}^d$ be Riemann-integrable. Prove the triangle inequality for $\mathbf{R}^d$-valued integrals:

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

(Hint: Let $y = \int_a^b f(x)\,dx$. If $y = 0$, then there is nothing to prove. Suppose $y \ne 0$. Let $u = y/|y|$, the unit vector in the direction of y. Then $|y| = \langle u \,|\, y \rangle = \left\langle u \,\middle|\, \int_a^b f(x)\,dx \right\rangle$. Check that $\left\langle u \,\middle|\, \int_a^b f(x)\,dx \right\rangle = \int_a^b \langle u \,|\, f(x) \rangle\,dx$ and then use Schwarz's inequality.)

Exercise 16. Let $f\colon [a,b] \to \mathbf{R}^d$ be Riemann-integrable and let $\varepsilon > 0$. Prove that there is a step function $\varphi\colon [a,b] \to \mathbf{R}$ such that $\int_a^b |f(x) - \varphi(x)|\,dx < \varepsilon$ and $|\varphi| \le |f|$.

A Review of Riemann Integration

Notation. Now let us extend an earlier notation. For each bounded function $f\colon [a,b] \to \mathbf{R}^d$, let

$$\rho(f) = R^*(|f|).$$

As before, $\rho(f)$ is defined even if f is not Riemann-integrable. However, if $f\colon [a,b] \to \mathbf{R}^d$ is Riemann-integrable, then $\rho(f) = \int_a^b |f(x)|\,dx$.

Exercise 17. Let $f\colon [a,b] \to \mathbf{R}^d$ be bounded. Suppose (f_n) is a sequence of Riemann-integrable functions $f_n\colon [a,b] \to \mathbf{R}^d$ such that $\rho(f - f_n) \to 0$. Prove that f is Riemann-integrable and that

$$\int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx.$$

Exercise 18. Let $f\colon [a,b] \to \mathbf{R}^d$ be Riemann-integrable and let $g\colon \mathbf{R}^d \to \mathbf{R}$ be continuous. Prove that $g \circ f$ is Riemann-integrable.

Riemann Integration of Complex-Valued Functions. Specializing the Riemann integral for $\mathbf{R}^d$-valued functions to the case where $d = 2$, we obtain the Riemann integral for complex-valued functions, by virtue of the usual identification between $\mathbf{R}^2$ and $\mathbf{C}$. Explicity, if $f\colon [a,b] \to \mathbf{C}$, then to say that f *is Riemann-integrable* (*over* $[a,b]$) means that $\operatorname{Re} f$ and $\operatorname{Im} f$, the real and imaginary parts of f, are both Riemann-integrable (over $[a,b]$) and in this case, we define *the Riemann integral of* f (*over* $[a,b]$) to be

$$\int_a^b f(x)\,dx = \int_a^b \operatorname{Re} f(x)\,dx + i \int_a^b \operatorname{Im} f(x)\,dx.$$

From our discussion of the Riemann integral for $\mathbf{R}^d$-valued functions, we already know that (4) and (5) hold for all Riemann-integrable functions $f, g\colon [a,b] \to \mathbf{C}$ and all $c \in \mathbf{R}$. It remains only to verify that (5) continues to hold for all $c \in \mathbf{C}$. This is straightforward and I'll leave it to you.