$d \in D$, the interval $\big(f(d-), f(d+)\big)$ contains a rational number q_d (make the obvious modification if d is an endpoint of I). If $d, d' \in D$ with $d < d'$, then $f(d+) \le f(d'-)$, so $q_d < q_{d'}$. Thus the map $d \mapsto q_d$ of D into $\mathbf{Q}$ is injective; since $\mathbf{Q}$ is countable, it follows that D is countable. ∎

It is left as one of the exercises at the end of this chapter to show that for any countable subset S of $\mathbf{R}$ there exists an increasing function on $\mathbf{R}$ whose discontinuity set is precisely S.

3.2 Two Fundamental Theorems

The next two theorems are of use in many situations. They will be generalized in a later chapter.

3.14 Theorem. *A continuous real-valued function on a closed bounded interval attains maximum and minimum values.*

Proof. Suppose $f : J \to \mathbf{R}$ is continuous, where $J = [a, b]$ $(a < b)$. Let $M = \sup\{f(x) : x \in J\}$. If $M = +\infty$, there exists for each $n \in \mathbf{N}$ some $x_n \in J$ such that $f(x_n) > n$. According to the Bolzano–Weierstrass theorem (Theorem 2.17) there exists a subsequence (x_{n_k}) of (x_n) which converges to some c; since $a \le x_n \le b$ for every n, we have $a \le c \le b$. According to Proposition 3.8(e), we have $f(c) = \lim f(x_{n_k})$, but this is impossible since $f(x_{n_k}) > n_k \to +\infty$. Thus $M < +\infty$. Now choose, for each $n \in \mathbf{N}$, $x_n \in J$ such that $f(x_n) > M - 1/n$. Since also $f(x_n) \le M$, we have $f(x_n) \to M$ as $n \to \infty$. The sequence (x_n) has a convergent subsequence (y_n). Then $\big(f(y_n)\big)$ is a subsequence of $(f(x_n))$, so $f(y_n) \to M$; but if $y_n \to c$, Proposition 3.8 assures us that $f(y_n) \to f(c)$. Thus $f(c) = M$. The proof that f attains a minimum value is similar, or can be deduced from what we have proved by considering the function $-f$. ∎

3.15 Theorem. *If f is continuous on the interval $[a, b]$, and $f(a) < y < f(b)$, or $f(a) > y > f(b)$), there exists x, with $a < x < b$, such that $f(x) = y$.*

Proof. We may assume that $f(a) < y < f(b)$. Let $E = \{t \in [a, b] : f(t) < y\}$, so E is a nonempty $(a \in E)$ subset of $[a, b]$. Let $x = \sup E$, so $x \in [a, b]$. For each n there exists $x_n \in E$ such that $x - 1/n < x_n \le x$. Thus $f(x_n) < y$ for every n. Since $x_n \to x$, we have (by Proposition 3.8(e)) $\lim f(x_n) = f(x)$, so $f(x) \le y$. But $f(b) > y$ implies (since f is continuous at b) that there exists $\delta > 0$ such that $f(t) > y$ for all t with $b - \delta < t \le b$. Thus $x < b$. Hence there exist $t_n \in J$ with $x < t_n$ and $\lim t_n = x$. Since $t_n > x$, we have $t_n \notin E$, i.e., $f(t_n) \ge y$, so $f(x) = \lim f(t_n) \ge y$. Thus $f(x) = y$. ∎