The next exercise generalizes problem X19.

X20. Let X be a topological space, let $\mathscr{B}$ be a filter base on X, and let $p \in X$. Prove that p is a cluster point for $\mathscr{B}$ in X if and only if there exists a filter base $\mathscr{C}$ on X such that $\mathscr{C}$ is finer than $\mathscr{B}$ and $\mathscr{C}$ converges to p.

Reminder. Let (x_n) be a sequence in $[-\infty, \infty]$. For each n, let

$$a_n = \inf_{m \geq n} x_m \qquad \text{and} \qquad b_n = \sup_{m \geq n} x_m.$$

Then $a_n \leq a_{n+1}$ and $b_n \geq b_{n+1}$ for all n. Hence each of (a_n) and (b_n) has a limit in $[-\infty, \infty]$. By definition,

$$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} a_n \qquad \text{and} \qquad \limsup_{n \to \infty} x_n = \lim_{n \to \infty} b_n.$$

Note that this is not the definition given in Rudin, *Principles of Mathemtical Analysis*, but I recommend that you use it because it is a more convenient working definition than the one in Rudin.

X21. Let (x_n) and (y_n) be bounded sequences of real numbers.

(a) Prove that for each n, we have

$$\sup_{m \geq n} (x_m + y_m) \leq \sup_{m \geq n} x_m + \sup_{m \geq n} y_m$$

and

$$\inf_{m \geq n} (x_m + y_m) \geq \inf_{m \geq n} x_m + \inf_{m \geq n} y_m.$$

(Warning: Do not "calculate" with sup or inf. Any inequalities you assert involving sup must be justified based on the definition of sup. Similarly for inf. Also, please do not use proof by contradiction when it is not needed and does not shorten the argument.)

(b) Prove that

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n.$$

and

$$\liminf(x_n + y_n) \geq \liminf x_n + \liminf y_n.$$

(c) Prove that

$$\liminf(x_n + y_n) \leq \limsup x_n + \liminf y_n \leq \limsup(x_n + y_n)$$

and

$$\liminf(x_n + y_n) \leq \liminf x_n + \limsup y_n \leq \limsup(x_n + y_n).$$

(Hint: These inequalities follow from the ones in part (b) by routine calculations. For instance, try applying the second inequality in part (b) to the sequences $(-x_n)$ and $(x_n + y_n)$ instead of (x_n) and (y_n) and see what you get.)

(d) Use parts (b) and (c) to prove that if (x_n) is convergent, then

$$\limsup(x_n + y_n) = \lim x_n + \limsup y_n$$

and

$$\liminf(x_n + y_n) = \lim x_n + \liminf y_n.$$

X22. Give an example of two bounded sequences in $\mathbf{R}$ for which all of the inequalities in parts (a), (b), and (c) of problem X21 are strict. (Hint: There is an example in which each sequence just repeats its first four terms over and over.)

X23. Let (x_n) be a sequence in $\mathbf{R}$. For each n, let $s_n = (x_1 + \cdots + x_n)/n$ be the average of $x_1, \ldots, x_n$. *due 4Th*
Let $a = \liminf x_n$, $A = \liminf s_n$, $B = \limsup s_n$, and $b = \limsup x_n$.

(a) Prove that $B \le b$. (Hint: To show that $B \le b$, it suffices to show that for each $c \in (b, \infty)$, we have $B \le c$. This is even valid when $b = \infty$, since in that case there is nothing to show anyway!)

Similarly, $a \le A$. (Or this can be deduced by applying part (a) to the sequence $(-x_n)$ instead of (x_n).)

(b) Let $L \in [-\infty, \infty]$. Deduce from part (a) that if $x_n \to L$, then $s_n \to L$ too.

X24. Give an example of a sequence (x_n) in $\mathbf{R}$ such that $\limsup x_n = \infty$ and $\liminf x_n = -\infty$ but $(x_1 + \cdots + x_n)/n \to 0$.

Definition. Let (x_n) be a sequence in $\mathbf{R}$ and let $L \in \mathbf{R}$. To say that (x_n) *is Cesáro convergent to* L means that $(x_1 + \cdots + x_n)/n \to L$.

Remark. By problem X23(b), ordinary convergence implies Cesáro convergence. By problem X24, the converse does not hold in general.

I hope the last few exercises have convinced you that the definitions for lim inf and lim sup given above are good working definitions. The next exercise, in combination with problem X19, shows that these definitions are equivalent to the corresponding ones given in Rudin, *Principles of Mathemtical Analysis*.

X25. Let (x_n) be a sequence in $[-\infty, \infty]$.

(a) Let $a = \liminf x_n$ and let $b = \limsup x_n$. Prove that a and b are cluster points for (x_n) in $[-\infty, \infty]$.

(b) Let c be a cluster point for (x_n) in $[-\infty, \infty]$. Prove that $a \le c \le b$.

Thus a is the smallest cluster point for (x_n) and b is the largest cluster point for (x_n).

Definition. Let $\mathscr{B}$ be a filter base on $[-\infty, \infty]$. Then

$$\liminf \mathscr{B} = \sup\{\inf B : B \in \mathscr{B}\} \qquad \text{and} \qquad \limsup \mathscr{B} = \inf\{\sup B : B \in \mathscr{B}\},$$

by definition.

Example. Let (x_n) be a sequence in $[-\infty, \infty]$ and let $\mathscr{B}$ be the filter base of tails of (x_n). Then clearly

$$\liminf \mathscr{B} = \liminf x_n \qquad \text{and} \qquad \limsup \mathscr{B} = \limsup x_n.$$

The next exercise is a generalization of problem X25.

X26. Let $\mathscr{B}$ be a filter base on $[-\infty, \infty]$.

(a) Let $a = \liminf \mathscr{B}$ and let $b = \limsup \mathscr{B}$. Prove that a and b are cluster points for $\mathscr{B}$ in $[-\infty, \infty]$.

(b) Let c be a cluster point for $\mathscr{B}$ in $[-\infty, \infty]$. Prove that $a \le c \le b$.

Thus a is the smallest cluster point for $\mathscr{B}$ and b is the largest cluster point for $\mathscr{B}$.

Definition. Let (X, d) be a metric space and let $\mathscr{B}$ be a filter base on X. To say that $\mathscr{B}$ *is Cauchy* means that for each $\varepsilon > 0$, there exists $B \in \mathscr{B}$ such that $\operatorname{diam}(B) < \varepsilon$.

Remark. Let (X, d) be a metric space, let (x_n) be a sequence in X, and let $\mathscr{B}$ be the filter base of tails of (x_n). Then clearly $\mathscr{B}$ is Cauchy if and only if (x_n) is Cauchy.

The result in the next exercise may be viewed as a refinement of problem X9 and may be proved by a similar method.

X27. Let (X, d) be a complete metric space, so that each Cauchy sequence in X converges. Prove that each Cauchy filter base on X converges.

X28. Let (a_n) be a sequence of strictly positive real numbers. According to Theorem 3.37 in Rudin, *Principles of Mathematical Analysis*, Third Edition, *do before MT*

$$\liminf \frac{a_{n+1}}{a_n} \le \liminf a_n^{1/n} \qquad \text{and} \qquad \limsup a_n^{1/n} \le \limsup \frac{a_{n+1}}{a_n}.$$

(a) Rudin gives a self-contained proof of these inequalities. Give a short alternative proof, based on problem X23.

(b) Compare the proof of Theorem 3.37 in Rudin with the solution of problem X23(a).