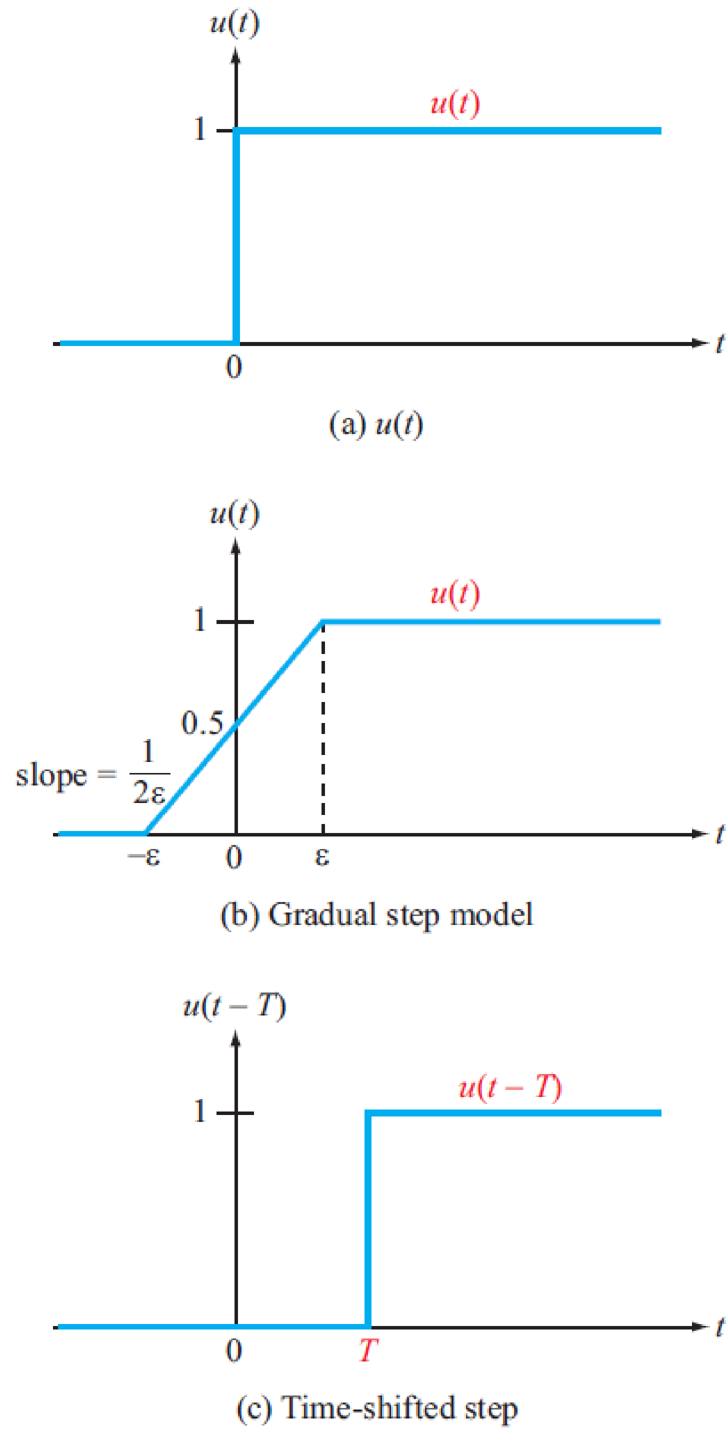
Singularity Functions

A ***singularity function*** is a function that either itself is not finite everywhere or one (or more) of its derivatives is (are) not finite everywhere.

Unit Step Function

$$u(t-T)=\begin{cases}0 & \text{for } t<T\\ 1 & \text{for } t>T\end{cases}$$



Singularity Functions (cont.)

Unit Impulse Function

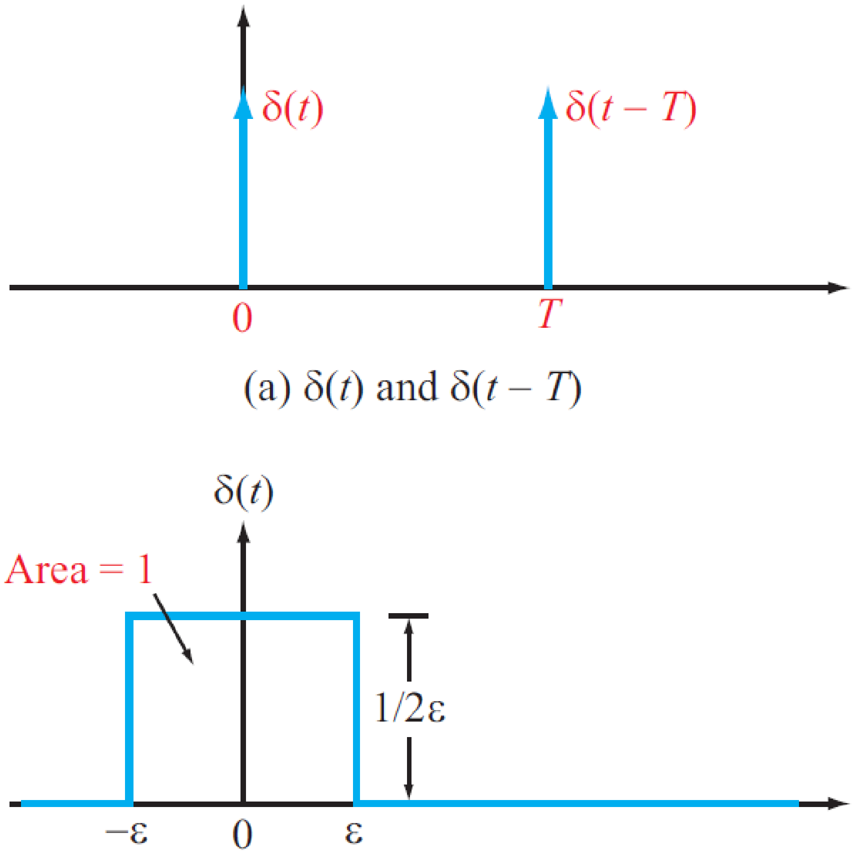
$$\delta(t-T) = 0 \qquad \text{for } t \neq T$$

$$\int_{-\infty}^{\infty} \delta(t-T)\, dt = 1.$$

$$\delta(t-T) = \frac{d}{dt} u(t-T) = u'(t-T).$$

For any function $f(t)$:

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-T)\, dt = f(T).$$



(a) δ(t) and δ(t – T)

(b) Rectangle model

Laplace Transform Definition

$$\mathbf{F}(\mathbf{s}) = \mathcal{L}[f(t)] = \int_{0^-}^{\infty} f(t)\, e^{-\mathbf{s}t}\, dt, \qquad (10.10)$$

where **s** is a complex variable with a real part σ and an imaginary part ω given by

$$\mathbf{s} = \sigma + j\omega. \qquad (10.11)$$

A given $f(t)$ has a unique $\mathbf{F}(\mathbf{s})$, and vice versa.

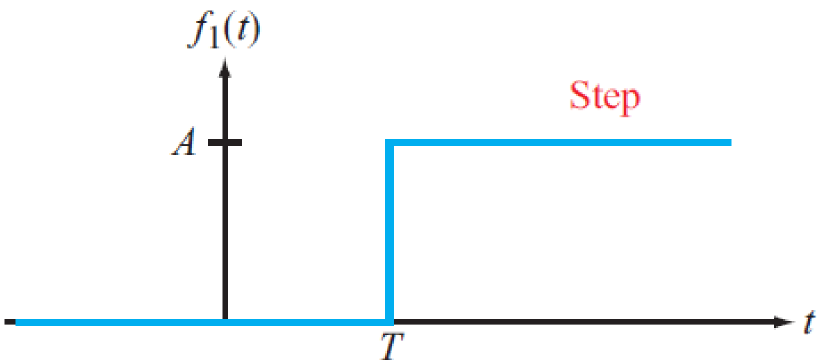
The uniqueness property can be expressed in symbolic form by

$$f(t) \quad \longleftrightarrow \quad \mathbf{F}(\mathbf{s}). \qquad (10.12a)$$

The two-way arrow is a shorthand notation for the combination of the two statements

$$\mathcal{L}[f(t)] = \mathbf{F}(\mathbf{s}), \qquad \text{and} \qquad \mathcal{L}^{-1}[\mathbf{F}(\mathbf{s})] = f(t). \qquad (10.12b)$$

Laplace Transform of Singularity Functions



$$f_1(t) = A\, u(t - T).$$

Application of Eq. (10.10) gives

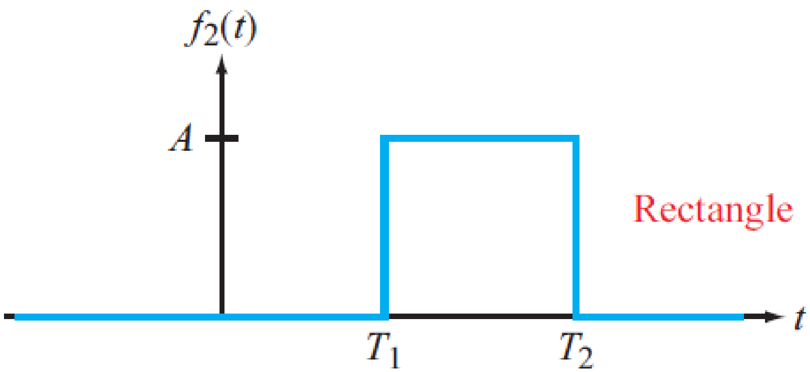
$$\mathbf{F}_1(\mathbf{s}) = \int_{0^-}^{\infty} f_1(t)\, e^{-\mathbf{s}t}\, dt = \int_{0^-}^{\infty} A\, u(t-T)\, e^{-\mathbf{s}t}\, dt$$

$$= A \int_{T}^{\infty} e^{-\mathbf{s}t}\, dt = -\frac{A}{\mathbf{s}}\, e^{-\mathbf{s}t}\Big|_{T}^{\infty} = \frac{A}{\mathbf{s}}\, e^{-\mathbf{s}T}.$$

$$f(t) \leftrightarrow \mathbf{F}(\mathbf{s})$$

For $A = 1$ and $T = 0$:

$$u(t) \leftrightarrow \frac{1}{\mathbf{s}}.$$



$$f_2(t) = A[u(t - T_1) - u(t - T_2)].$$

Its Laplace transform is

$$\mathbf{F}_2(\mathbf{s}) = \int_{0^-}^{\infty} A[u(t-T_1) - u(t-T_2)]e^{-\mathbf{s}t}\, dt$$

$$= A \int_{0^-}^{\infty} u(t-T_1)\, e^{-\mathbf{s}t}\, dt - A \int_{0^-}^{\infty} u(t-T_2)\, e^{-\mathbf{s}t}\, dt$$

$$= \frac{A}{\mathbf{s}}[e^{-\mathbf{s}T_1} - e^{-\mathbf{s}T_2}].$$

Laplace Transform of Delta Function

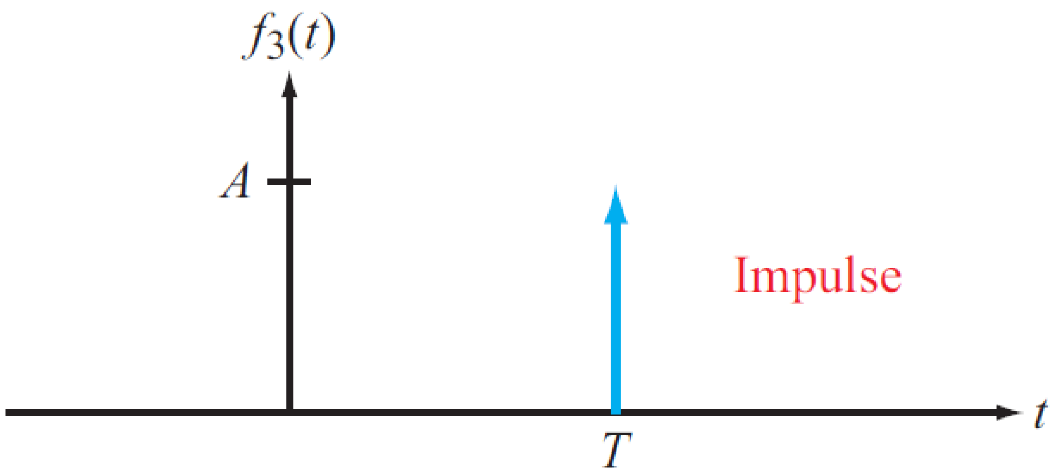
$$f_3(t) = A\,\delta(t - T).$$

The corresponding Laplace transform is

$$\mathbf{F}_3(\mathbf{s}) = \int_{0^-}^{\infty} A\,\delta(t - T)\,e^{-\mathbf{s}t}$$

$$= A \int_{T-\epsilon}^{T+\epsilon} \delta(t - T)\,e^{-\mathbf{s}t}\,dt$$

$$= Ae^{-\mathbf{s}T},$$



$f(t) \leftrightarrow \mathbf{F}(\mathbf{s})$

For $A = 1$ and $T = 0$: $\delta(t) \leftrightarrow 1.$

Example 10-3: Laplace Transform of $\cos \omega t$

Obtain the Laplace transform for $[\cos \omega t]\, u(t)$.

Solution: The solution is facilitated by expressing $\cos \omega t$ in terms of complex exponentials (see Appendix D-1), namely

$$\cos \omega t = \frac{1}{2}[e^{j\omega t} + e^{-j\omega t}].$$

Use of this expression in Eq. (10.10) gives

$$\begin{aligned}
\mathbf{F}(\mathbf{s}) &= \int_{0^-}^{\infty} \cos \omega t \; u(t) \; e^{-\mathbf{s}t} \; dt \\
&= \frac{1}{2}\left[\int_{0}^{\infty} e^{j\omega t} e^{-\mathbf{s}t} \; dt + \int_{0}^{\infty} e^{-j\omega t} e^{-\mathbf{s}t} \; dt\right] \\
&= \frac{1}{2}\left[\frac{e^{(j\omega - \mathbf{s})t}}{j\omega - \mathbf{s}} + \frac{e^{-(j\omega + \mathbf{s})t}}{-(j\omega + \mathbf{s})}\right]\Bigg|_{0}^{\infty} = \frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}.
\end{aligned}$$

Hence,

$$\cos \omega t \quad \longleftrightarrow \quad \frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}. \qquad (10.17)$$

Properties of Laplace Transform

1. Time Scaling

If

$$f(t) \leftrightarrow \mathbf{F}(\mathbf{s}),$$

then the transform of the time-scaled function $f(at)$ is

$$f(at) \leftrightarrow \frac{1}{a}\mathbf{F}\left(\frac{\mathbf{s}}{a}\right), \qquad a > 0.$$

2. Time Shift

If t is shifted by T along the time axis with $T \geq 0$, then

$$f(t-T)\, u(t-T) \leftrightarrow e^{-T\mathbf{s}}\, \mathbf{F}(\mathbf{s}), \quad T \geq 0.$$

Example

$$\cos \omega t \leftrightarrow \frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}.$$

$$\cos \omega(t-T)\, u(t-T) \leftrightarrow e^{-T\mathbf{s}} \frac{\mathbf{s}}{\mathbf{s}^2 + \omega^2}.$$

Properties of Laplace Transform (cont.)

3. Frequency Shift

$$e^{-at} f(t) \longleftrightarrow \mathbf{F}(\mathbf{s}+a).$$

4. Time Differentiation

$$f' = \frac{df}{dt} \longleftrightarrow \mathbf{s}\,\mathbf{F}(\mathbf{s}) - f(0^-).$$

$$f'' = \frac{d^2 f}{dt^2} \longleftrightarrow \mathbf{s}^2\,\mathbf{F}(\mathbf{s}) - \mathbf{s}\,f(0^-) - f'(0^-),$$

Example

For $f(t) = \cos \omega t$,

$$\mathbf{F}(\mathbf{s}) = \frac{\mathbf{s}}{\mathbf{s}^2+\omega^2},$$

$$f(0^-) = 1,$$

and

$$f'(0^-) = -\omega \sin \omega t|_{t=0^-} = 0.$$

Hence,

$$\mathcal{L}[f''] = \frac{\mathbf{s}^3}{\mathbf{s}^2+\omega^2} - \mathbf{s} = \frac{-\omega^2\mathbf{s}}{\mathbf{s}^2+\omega^2}.$$

Properties of Laplace Transform (cont.)

5. Time Integration

$$\int_0^t f(t)\, dt \quad \longleftrightarrow \quad \frac{1}{\mathbf{s}} \mathbf{F}(\mathbf{s}).$$

6. Initial and Final-Value Theorems

$$f(0^+) = \lim_{\mathbf{s}\to\infty} \mathbf{s}\, \mathbf{F}(\mathbf{s})$$

Initial-value theorem,

$$f(\infty) = \lim_{\mathbf{s}\to 0} \mathbf{s}\, \mathbf{F}(\mathbf{s})$$

Final-value theorem.

Example 10-5:
Initial and Final Values

Determine the initial and final values of a function $f(t)$ whose Laplace transform is given by

$$\mathbf{F}(\mathbf{s}) = \frac{25\mathbf{s}(\mathbf{s}+3)}{(\mathbf{s}+1)(\mathbf{s}^2+2\mathbf{s}+36)}.$$

Solution: Application of Eq. (10.39) gives

$$f(0^+) = \lim_{\mathbf{s}\to\infty} \mathbf{s}\,\mathbf{F}(\mathbf{s}) = \lim_{\mathbf{s}\to\infty} \frac{25\mathbf{s}^2(\mathbf{s}+3)}{(\mathbf{s}+1)(\mathbf{s}^2+2\mathbf{s}+36)}.$$

To avoid the problem of dealing with ∞, it often is more convenient to first apply the substitution $\mathbf{s} = 1/\mathbf{u}$, rearrange the function and then to find the limit as $\mathbf{u} \to 0$. That is,

$$\begin{aligned} f(0^+) &= \lim_{\mathbf{u}\to 0} \frac{25(1/\mathbf{u}+3)}{\mathbf{u}^2(1/\mathbf{u}+1)(1/\mathbf{u}^2+2/\mathbf{u}+36)} \\ &= \lim_{\mathbf{u}\to 0} \frac{25(1+3\mathbf{u})}{(1+\mathbf{u})(1+2\mathbf{u}+36\mathbf{u}^2)} \\ &= \frac{25(1+0)}{(1+0)(1+0+0)} = 25. \end{aligned}$$

To determine $f(\infty)$, we apply Eq. (10.40),

$$\begin{aligned} f(\infty) &= \lim_{\mathbf{s}\to 0} \mathbf{s}\,\mathbf{F}(\mathbf{s}) \\ &= \lim_{\mathbf{s}\to 0} \frac{25\mathbf{s}^2(\mathbf{s}+3)}{(\mathbf{s}+1)(\mathbf{s}^2+2\mathbf{s}+36)} = 0. \end{aligned}$$

Properties of Laplace Transform (cont.)

7. Frequency Differentiation

$$t\ f(t) \leftrightarrow -\frac{d\ \mathbf{F}(\mathbf{s})}{d\mathbf{s}} = -\mathbf{F}'(\mathbf{s}),$$

8. Frequency Integration

$$\frac{f(t)}{t} \leftrightarrow \int_{\mathbf{s}}^{\infty} \mathbf{F}(\mathbf{s})\ d\mathbf{s}.$$

Example 10-6: Applying Frequency Differentiation Property

Given that

$$\mathbf{F}(\mathbf{s}) = \mathcal{L}[e^{-at}] = \frac{1}{\mathbf{s}+a},$$

apply Eq. (10.43) to obtain the Laplace transform of te^{-at}.

Solution:

$$\mathcal{L}[te^{-at}] = -\frac{d}{d\mathbf{s}}\ \mathbf{F}(\mathbf{s}) = -\frac{d}{d\mathbf{s}}\left[\frac{1}{\mathbf{s}+a}\right]$$
$$= \frac{1}{(\mathbf{s}+a)^2}.$$

Table 10-1: Properties of the Laplace transform.

Property	$f(t)$		$\mathbf{F}(\mathbf{s}) = \mathcal{L}[f(t)]$
1. Multiplication by constant	$K\ f(t)$	$\Leftrightarrow$	$K\ \mathbf{F}(\mathbf{s})$
2. Linearity	$K_1\ f_1(t) + K_2\ f_2(t)$	$\Leftrightarrow$	$K_1\ \mathbf{F}_1(\mathbf{s}) + K_2\ \mathbf{F}_2(\mathbf{s})$
3. Time scaling	$f(at), \quad a > 0$	$\Leftrightarrow$	$\frac{1}{a}\ \mathbf{F}\left(\frac{\mathbf{s}}{a}\right)$
4. Time shift	$f(t-T)\ u(t-T)$	$\Leftrightarrow$	$e^{-T\mathbf{s}}\ \mathbf{F}(\mathbf{s})$
5. Frequency shift	$e^{-at}\ f(t)$	$\Leftrightarrow$	$\mathbf{F}(\mathbf{s}+a)$
6. Time 1st derivative	$f' = \frac{df}{dt}$	$\Leftrightarrow$	$\mathbf{s}\ \mathbf{F}(\mathbf{s}) - f(0^-)$
7. Time 2nd derivative	$f'' = \frac{d^2 f}{dt^2}$	$\Leftrightarrow$	$\mathbf{s}^2\mathbf{F}(\mathbf{s}) - \mathbf{s} f(0^-) - f'(0^-)$
8. Time integral	$\int_0^t f(t)\ dt$	$\Leftrightarrow$	$\frac{1}{\mathbf{s}}\ \mathbf{F}(\mathbf{s})$
9. Frequency derivative	$t\ f(t)$	$\Leftrightarrow$	$-\frac{d}{d\mathbf{s}}\ \mathbf{F}(\mathbf{s}) = -\mathbf{F}'(\mathbf{s})$
10. Frequency integral	$\frac{f(t)}{t}$	$\Leftrightarrow$	$\int_{\mathbf{s}}^{\infty} \mathbf{F}(\mathbf{s})\ d\mathbf{s}$
11. Initial value	$f(0^+)$	=	$\lim_{\mathbf{s}\to\infty} \mathbf{s}\ \mathbf{F}(\mathbf{s})$
12. Final value	$f(\infty)$	=	$\lim_{\mathbf{s}\to 0} \mathbf{s}\ \mathbf{F}(\mathbf{s})$
13. Convolution	$f_1(t) * f_2(t)$	$\Leftrightarrow$	$\mathbf{F}_1(\mathbf{s})\ \mathbf{F}_2(\mathbf{s})$

Table 10-2: Examples of Laplace transform pairs. Note that $f(t) = 0$ for $t < 0^-$.

	Laplace Transform Pairs		
	$f(t)$		$F(s) = \mathcal{L}[f(t)]$
1	$\delta(t)$	$\Leftrightarrow$	1
1a	$\delta(t-T)$	$\Leftrightarrow$	e^{-Ts}
2	$u(t)$	$\Leftrightarrow$	$\frac{1}{s}$
2a	$u(t-T)$	$\Leftrightarrow$	$\frac{e^{-Ts}}{s}$
3	$e^{-at}\,u(t)$	$\Leftrightarrow$	$\frac{1}{s+a}$
3a	$e^{-a(t-T)}\,u(t-T)$	$\Leftrightarrow$	$\frac{e^{-Ts}}{s+a}$
4	$t\,u(t)$	$\Leftrightarrow$	$\frac{1}{s^2}$
4a	$(t-T)\,u(t-T)$	$\Leftrightarrow$	$\frac{e^{-Ts}}{s^2}$
5	$t^2\,u(t)$	$\Leftrightarrow$	$\frac{2}{s^3}$
6	$te^{-at}\,u(t)$	$\Leftrightarrow$	$\frac{1}{(s+a)^2}$
7	$t^2e^{-at}\,u(t)$	$\Leftrightarrow$	$\frac{2}{(s+a)^3}$
8	$t^{n-1}e^{-at}\,u(t)$	$\Leftrightarrow$	$\frac{(n-1)!}{(s+a)^n}$
9	$\sin\omega t\,u(t)$	$\Leftrightarrow$	$\frac{\omega}{s^2+\omega^2}$
10	$\sin(\omega t+\theta)\,u(t)$	$\Leftrightarrow$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
11	$\cos\omega t\,u(t)$	$\Leftrightarrow$	$\frac{s}{s^2+\omega^2}$
12	$\cos(\omega t+\theta)\,u(t)$	$\Leftrightarrow$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
13	$e^{-at}\sin\omega t\,u(t)$	$\Leftrightarrow$	$\frac{\omega}{(s+a)^2+\omega^2}$
14	$e^{-at}\cos\omega t\,u(t)$	$\Leftrightarrow$	$\frac{s+a}{(s+a)^2+\omega^2}$
15	$2e^{-at}\cos(bt-\theta)\,u(t)$	$\Leftrightarrow$	$\frac{e^{j\theta}}{s+a+jb}+\frac{e^{-j\theta}}{s+a-jb}$
16	$\frac{2t^{n-1}}{(n-1)!}e^{-at}\cos(bt-\theta)\,u(t)$	$\Leftrightarrow$	$\frac{e^{j\theta}}{(s+a+jb)^n}+\frac{e^{-j\theta}}{(s+a-jb)^n}$

Example 10-7: Laplace Transform

Obtain the Laplace transform of $f(t) = t^2 e^{-3t} \cos 4t$.

Solution: The given function is a product of three functions. We start with the cosine function which we will call $f_1(t)$:

$$f_1(t) = \cos 4t. \tag{10.47}$$

According to entry #11 in Table 10-2, the corresponding Laplace transform is

$$\mathbf{F}_1(\mathbf{s}) = \frac{\mathbf{s}}{\mathbf{s}^2 + 16}. \tag{10.48}$$

Next, we define

$$f_2(t) = e^{-3t} \cos 4t = e^{-3t} \; f_1(t), \tag{10.49}$$

and we apply the frequency-shift property (entry #5 in Table 10-1) to obtain

$$\mathbf{F}_2(\mathbf{s}) = \mathbf{F}_1(\mathbf{s}+3) = \frac{\mathbf{s}+3}{(\mathbf{s}+3)^2 + 16}, \tag{10.50}$$

where we replaced $\mathbf{s}$ with $(\mathbf{s}+3)$ everywhere in the expression of Eq. (10.48). Finally, we define

$$f(t) = t^2 \; f_2(t) = t^2 e^{-3t} \cos 4t, \tag{10.51}$$

and we apply the frequency derivative property (entry #9 in Table 10-1) twice:

$$\begin{aligned}\mathbf{F}(\mathbf{s}) = \mathbf{F}_2''(\mathbf{s}) &= \frac{d^2}{d\mathbf{s}^2}\left[\frac{\mathbf{s}+3}{(\mathbf{s}+3)^2+16}\right] \\ &= \frac{2(\mathbf{s}+3)[(\mathbf{s}+3)^2 - 48]}{[(\mathbf{s}+3)^2+16]^3}.\end{aligned} \tag{10.52}$$