APPENDIX A

PRINCIPAL CLASSES OF FUNCTIONS AND SETS

N is the set of all nonnegative integers. |x| is max(x).

MF is the set of all functions whose domain is a subset of some N^k and whose range is a subset of N.

SD is the set of all f ∈ MF such that for all x ∈ dom(f), f(x) > |x|.

EVSD is the set of all f ∈ MF such that for all but finitely many x ∈ dom(f), f(x) > |x|.

ELG is the set of all f ∈ MF such that there exist c,d > 1 obeying the following condition. For all but finitely many x ∈ dom(f), c|x| ≤ f(x) ≤ d|x|.

LB is the set of all f ∈ MF such that there exists d obeying the following condition. For all x ∈ dom(f), |x| ≤ d|x|.

EXPN is the set of all f ∈ MF such that there exists c > 1 obeying the following condition. For all but finitely many x ∈ dom(f), c|x| ≤ f(x).

BAF is the set of all f ∈ MF which can be written using 0,1,+,-,•,↑,log, where x-y = max(x-y,0), x↑ = 2^x, log(x) = floor(log(x)) if x > 0; 0 otherwise. Closure under definition by cases, using <,=, is derived in section 5.1.

INF is the set of all infinite subsets of N.

PRINCIPAL FORMAL SYSTEMS

The systems RCA_0, WKL_0, ACA_0 , ATR_0, Π^1_1-CA_0 of Reverse Mathematics (see section 0.4).

The systems ACA', ACA^+ (Definitions 1.4.1, 6.2.1).

The systems ZFC, MAH, SMAH, MAH^+, $SMAH^+$. MAH = ZFC + {there exists an n-Mahlo cardinal$\}_n$. SMAH = ZFC + {there exists a strongly n-Mahlo cardinal$\}_n$. MAH^+ = ZFC + $(\forall n < \omega)(\exists\kappa)(\kappa$ is an n-Mahlo cardinal). $SMAH^+$ = ZFC + $(\forall n < \omega)(\exists\kappa)(\kappa$ is a strongly n-Mahlo cardinal). (Definitions 4.1.1, 4.1.2).

INDEPENDENT PROPOSITIONS

PROPOSITION A. For all f,g $\in$ ELG there exist A,B,C $\in$ INF such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

PROPOSITION B. Let f,g $\in$ ELG and n $\geq$ 1. There exist infinite $A_1 \subseteq ... \subseteq A_n \subseteq N$ such that
i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii) $A_1 \cap fA_n = \emptyset$.

PROPOSITION C. For all f,g $\in$ ELG $\cap$ SD $\cap$ BAF, there exist A,B,C $\in$ INF such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

$\cup.$ is disjoint union. Its presence indicates that its left and right sides are disjoint sets.

Trivial implications: B $\rightarrow$ A $\rightarrow$ C.

Proposition A is the Principal Exotic Case, which arises in Chapter 3 (see section 3.13). Proposition B is proved in Chapter 4 in ACA' + 1-Con(SMAH). Proposition C is shown in Chapter 5 to imply 1-Con(SMAH) in ACA'.

In section 6.1, we treat the following five Propositions.

PROPOSITION D. Let f $\in$ LB $\cap$ EVSD, g $\in$ EXPN, E $\subseteq$ N be infinite, and n $\geq$ 1. There exist infinite $A_1 \subseteq ... \subseteq A_n \subseteq N$ such that
i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii) $A_1 \cap fA_n = \emptyset$;
iii) $A_1 \subseteq E$.

PROPOSITION E. For all f,g $\in$ ELG $\cap$ SD $\cap$ BAF there exist A $\subseteq$ B $\subseteq$ C $\subseteq$ N, each containing infinitely many powers of 2, such that

$$fA \subseteq B \cup. gB$$

$$fB \subseteq C \cup. gC$$

PROPOSITION F. For all f,g ∈ ELG ∩ SD ∩ BAF there exist A ⊆ B ⊆ C ⊆ N, each containing infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$

PROPOSITIOIN G. For all f,g ∈ ELG ∩ SD ∩ BAF there exist A,B,C ⊆ N, whose intersection contains infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$

PROPOSITIOIN H. For all f,g ∈ ELG ∩ SD ∩ BAF there exist A,B,C ⊆ N, where A ∩ B contains infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$

Each of these seven Propositions are shown in ACA' to be equivalent to 1-Con(SMAH).

Trivial implications: D → B → A → C, and D → E → F → G → H.

In section 6.2, we treat the following arithmetic forms.

PROPOSITION C[prim]. For all f,g ∈ ELG ∩ SD ∩ BAF, there exist A,B,C ∈ INF with primitive recursive enumeration functions, such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

PROPOSITION E[prim]. For all f,g ∈ ELG ∩ SD ∩ BAF there exist A ⊆ B ⊆ C ⊆ N with primitive recursive enumeration functions, each containing infinitely many powers of 2, such that

$$fA \subseteq B \cup. gB$$
$$fB \subseteq C \cup. gC$$

PROPOSITION F[prim]. For all f,g ∈ ELG ∩ SD ∩ BAF there exist A ⊆ B ⊆ C ⊆ N with primitive enumeration functions, each containing infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$

PROPOSITION G[prim]. For all $f,g \in ELG \cap SD \cap BAF$ there exist $A,B,C \subseteq N$ with primitive recursive enumeration functions, whose intersection contains infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$

PROPOSITION H[prim]. For all $f,g \in ELG \cap SD \cap BAF$ there exist $A,B,C \subseteq N$ with primitive enumeration functions, where $A \cap B$ contains infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$