1.2. Some BRT settings.

The BRT settings were defined in Definition 1.11.

Most areas of mathematics have a naturally associated family of (multivariate) functions and sets. This usually forms a natural and interesting BRT setting.

This book focuses on five basic BRT settings, as noted in section 1.1. These are formally introduced in Chapter 2. In fact, we have only been able to scratch the surface of BRT even on these basic BRT settings.

In this section, we survey a huge range of mathematically interesting BRT settings. This will give the reader a sense of the unusual scope of BRT and a glimpse of what can be expected in the future development of BRT.

We provide a plausible estimate that at least 1,000,000 of these mathematically interesting BRT settings represent significantly different BRT phenomena. Any substantial probing of BRT on these settings is beyond the scope of this book.

In sections 1.3 and 1.4, we will investigate the status of the Complementation Theorem and the Thin Set Theorem in some very modest sampling of the BRT settings presented in this section. This will give a modest indication as to the depth of BRT and its sensitivity to the choice of BRT setting.

I. On N.

We now consider a number of natural conditions on functions in MF. These conditions are of three kinds.

1. Bounding conditions.
2. Regularity conditions.
3. Choice of norm.

We propose the following basic lower bound conditions on f ∈ MF.

i. There exist c,d such that c op i, d op′ j, and for all x ∈ dom(f), $c|x|^d$ op′′ f(x). Here op,op′ ∈ {<,>,≤,≥,=}, op′′ ∈ {<,≤}, i,j ∈ {0,1/2,1,3/2,2}, and | | is the l_∞ norm, the l_1 norm, or the l_2 norm.

We propose the following basic upper bound conditions on f ∈ MF.

ii. There exist c,d such that c op i, d op′ j, such that for all x ∈ dom(f), $c|x|^d$ op′′ f(x). Here op,op′ ∈ {<,>,≤,≥,=}, op′′ ∈ {>,≥}, i,j ∈ {0,1/2,1,3/2,2}, and | | is the l_∞ norm, the l_1 norm, or the l_2 norm.

Each of these conditions in i,ii above results from the choice of 6 parameters: op,op',op'',i,j,| |. Note that some of the choices of parameters result in degenerate conditions.

Each of these basic lower bound conditions and basic upper bound conditions can be modified by using "for all but finitely many x" instead of "for all x". This doubles the number of lower and upper conditions, and the resulting conditions are called the lower bound conditions and the upper bound conditions.

The bounding conditions consist of a conjunction of zero or more conditions, each of which is either a basic lower bound condition or a basic upper bound condition.

The five basic BRT settings formally introduced in Chapter 2 are examples of classes of functions obeying bounding conditions.

The basic regularity conditions that we propose on f ∈ MF are as follows.

i. f is a linear function.
ii. f is a polynomial of degree op d with integer (or rational) coefficients. Here op ∈ {=,≤,≥} and d ∈ {1,2,3,4}.
iii. f is a polynomial with integer (or rational) coefficients.
iv. f is given by an expression in some chosen subset of the operations 0,1,+,−,•,÷,↑,log,exp.
v. f is a Presburger function; i.e., definable in (N,+).
vi. f is a primitive recursive function.
vii. f is a recursive function.
viii. f is an arithmetic function.
ix. f is a hyperarithmetic function.

The class of functions BAF = EBAF introduced in section 5.1 is an example of a set of functions obeying a basic regularity condition (see iv above). In Definition 5.1.1, we define 0,1,+,-,•,↑,log used in iv above. Here exp(n,m) = n^m, where exp(n,0) = 1. Also x÷y is the floor of (x divided by y), where x÷0 is taken to be 0.

We place two modifiers on the basic regularity conditions. One is that we allow finitely many exceptions in conditions i - iv above. The second is that we allow conditions i - iv above to merely hold piecewise. I.e., we modify each of conditions i - iv above, to assert only that the function obeys the condition on each of finitely many pieces, where each piece is given by a finite set of inequalities involving functions obeying that same condition. Note that these modifiers have no effect on conditions v - ix above.

Finally, the conditions on f ∈ MF that we propose consist of the conjunction of zero or more conditions, each of which are either a bounding condition, or a regularity condition (perhaps modified).

We now come to the proposed conditions on A ∈ INF.

Firstly, we have the density conditions.

i. A has (upper, lower) density d ∈ ‘a,b’, where a ≤ b, a,b ∈ {0,1/3,1/2,2/3,1}, and ‘ is (or [, and ’ is) or].

Secondly, we have the arithmetic and geometric progression conditions.

ii. A is (contains, is contained in, excludes) an arithmetic (geometric) progression.
iii. Same as ii) with “eventually”.
iv. A meets every arithmetic (geometric) progression.
v. A contains infinitely many even (odd) elements.
vi. A has arbitrarily long consecutive runs.

Thirdly, we have the regularity conditions.

vii. A is a Presburger set.
viii. A is a primitive recursive set.
ix. A is a recursive set.
x. A is an arithmetic set.
xi. A is a hyperarithmetic set.

Finally, the conditions on $A \in INF$ that we propose consist of zero or more conditions, each of which is either a density condition, an arithmetic/geometric progression condition, or a regularity condition.

The BRT settings that we propose on N are the (V,K), where V is the set of all $f \in MF$ obeying a condition in this part I, and K is the set of all $A \in INF$ obeying a condition in this part I.

II. On **Z**.

The conditions on functions proposed in part I on functions from MF have natural counterparts as conditions on functions from MF(**Z**) = the class of all multivariate functions from **Z** into **Z**. Specifically, in the bounding conditions, we use $|f(x)|$ instead of $f(x)$. The basic regularity conditions extend to MF(**Z**) in obvious natural ways.

Let INF(**Z**) be the set of all infinite subsets of **Z**. Every pair of conditions α,β on INF from part I generates conditions on $A \in INF(\mathbf{Z})$ as follows.

i. α holds of $A \cap N$.
ii. β holds of $-A \cap N$.
iii. α holds of $A \cap N$ and β holds of $-A \cap N$.

Here are three particularly natural examples of such conditions on $A \in INF(\mathbf{Z})$.

i. No condition. I.e., $A \in INF(\mathbf{Z})$.
ii. A^+ is infinite.
iii. A^+ and A^- are infinite.

III. On **Q**.

We lift the conditions on **Z** to **Q**. We can additionally use "for all arguments of sufficiently large norm", in addition to "for all but finitely many".

We can also place Lipshitz conditions everywhere, or merely for all arguments of sufficiently large norm. We can restrict attention to Lipshitz conditions involving $c|x-y|^d$, where the constants c,d are treated in a manner similar to the way c,d were treated in connection with lower (and upper) bounds in A above.

In the regularity conditions on functions, Presburger can be replaced by "definable in the ordered additive group of rationals", or "definable in the ordered additive group of rationals with a predicate for the integers".

The density and arithmetic/geometric progression conditions on sets can be replaced by conditions involving the Jordan content of the subset of Q.

IV. On $\Re$.

We can lift the conditions on **Q** to $\Re$. We can additionally add pointwise continuity, uniform continuity, differentiability, and real analyticity conditions on the functions.

We can use semialgebraic as a regularity condition on sets. We can also use open, closed, F_σ, and G_δ as regularity conditions. We can use Lebesgue measure instead of Jordan content.

V. On **C**.

We can of course treat the complex plane **C** like $\Re^2$ and lift the conditions on $\Re$ to conditions on **C**. But it is interesting to use analyticity and other important notions from complex analysis as regularity conditions.

VI. On L^2.

Here we should focus attention on the set V of all bounded linear operators on L^2, and the set K of all nontrivial closed subspaces of L^2. The invariant subspace problem for L^2 is expressed as the followings instance of EBRT in A,fA on (V,K):

$$(\forall f \in V)(\exists A \in K)(fA = A).$$

We can obviously use other function spaces for BRT settings.

VII. Topological BRT.

Here we can extend the conditions used on $\Re$ above to various specific topological spaces. Being nonempty and open is a particularly natural condition on sets.

It also makes sense to investigate those BRT statements that hold in the continuous functions and nonempty open sets, on all topological spaces obeying certain conditions.

The above gives a mere indication of just some of the wide ranging natural BRT settings throughout mathematics.

We believe that the truth value of BRT statements depend very delicately on the choice of BRT setting. In fact, we believe that this delicate relationship already manifests itself in EBRT and IBRT in A,B,C,fA,fB,fC,gA,gB,gC.

Indications of this sensitivity are already present in our classifications of Chapter 2 as well as our results in section 1.4.

In particular, we expect that for many pairs of settings presented in parts I-VII, both the EBRT and the IBRT statements in A,B,C,fA,fB,fC,gA,gB,gC differ. In fact, we suspect that this is true even for small fragments of A,B,C,fA,fB,fC,gA,gB,gC.

We now give a very crude lower estimate on the number of settings presented in parts I-VII above, that we suspect have different BRT behavior. Note that we have been fully precise only in part I above concerning BRT settings on N. So we will only make a rough lower estimate of the number of specific BRT settings proposed on N.

This will not take into account the greater richness that comes with working on other underlying sets, which are generally endowed with various structure, as in parts II-VII above.

In the basic lower bound conditions, there are 5 choices of op, 5 choices of op', 2 choices of op'', 5 choices of i, 5 choices of j, and 3 choices of norm. This results in 5 x 5 x 2 x 5 x 5 x 3 = 3750.

A conservative lower estimate as to the number of these lower bound conditions that are substantially different for BRT purposes is sqrt(3750), which is approximately 60.

Analogously, a conservative lower estimate for basic upper bound conditions is also 60.

The choice of going from "for all x" to "for all but finitely many x" should double these numbers to 120. Conjuncting lower bound conditions and upper bound conditions could result in at least 120 x 120 = 14400 bounding conditions. The interactions between the lower and upper bound conditions might not be all that strong, and so a conservative lower estimate for the bounding conditions for our purposes is 1000.

There are 24 basic regularity conditions under ii above, in addition to 8 others. For BRT significance, we use the conservative estimate of 10.

We have the two modifiers - finitely many exceptions and piecewiseness. This replaces 10 by 20, from a conservative point of view.

Finally, the interaction of bounding conditions and regularity conditions should conservatively result in an estimate of 10,000 families of multivariate functions on N which are substantially different for BRT purposes.

We now take the conditions on subsets of N into consideration.

The density conditions have the upper/lower parameter, 15 pairs a,b, and 4 choices of kinds of intervals. This results in 2 x 15 x 4 = 120 possibilities. Conservatively, we use the lower estimate of 25 for our purposes.

In the arithmetic and geometric progression conditions, items ii and iii each have 6 possibilities, items iv,v each have 2 possibilities, and one for vi, for a total of 17. We use the lower estimate of 8 for our purposes.

There are 5 regularity conditions on sets. We conservatively estimate that 2 have substantial BRT significance.

This results in a triple product 25 x 8 x 2 = 400. We use the conservative lower estimate of 100 for conditions on sets with substantial BRT significance.

We have been sufficiently conservative and believe that our lower estimates for the conditions on multivariate functions, and conditions on sets, are rather lean and mean. Hence for our final estimate, we will simply multiply

10,000 by 100. Thus our lower estimate on the number of interesting BRT settings on N that have been presented, that are BRT different in A,B,C,fA,fB,fC,gA,gB,gC, is 1,000,000.

It is beyond the scope of this book to provide substantive justification for this conjectured lower estimate.