1.4. Thin Set Theorems.

Recall the Thin Set Theorem from section 1.1.

THIN SET THEOREM. For all $f \in MF$ there exists $A \in INF$ such that $fA \neq N$.

The Thin Set Theorem as written above is a statement of IBRT in A,fA on (MF,INF). This specific statement is due to the present author, who studied it for its significance for Reverse Mathematics and recursion theory.

A variant of this statement was already introduced much earlier in the literature on the (square bracket) partition calculus in combinatorial set theory, in [EHR65]. In their language, the Thin Set Theorem reads

$$(\forall n < \omega)(\omega \rightarrow [\omega]^n_\omega).$$

The n indicates a coloring of the unordered n-tuples from the ω to the left of $\rightarrow$, the lower ω indicates the number of colors, and the ω in [ ] indicates the cardinality of the "homogenous" set. But here [ ] indicates a weak form of homogeneity - that at least one color is omitted.

The mathematical difference between this square bracket partition relation statement and the Thin Set Theorem is that the former involves unordered tuples, whereas the latter involves ordered tuples. However, see Theorem 1.4.2 below for an equivalence proof in RCA_0. Also see [EHMR84], Theorem 54.1. It was immediately recognized that this square bracket partition relation follows from the usual infinite Ramsey theorem, which is written in terms of the round parenthesis partition relation

$$(\forall n,m < \omega)(\omega \rightarrow (\omega)^n_m).$$

Experience reveals that when the Thin Set Theorem is stated exactly in our formulation above (with ordered n tuples), mathematicians who are not experts in the partition calculus, do not recognize the Thin Set Theorem's connection with the partition calculus and combinatorial set theory. They are struck by its fundamental character, and will not be able to prove it in short order. They apparently would have to rediscover the infinite Ramsey theorem, and in our experience, long before they invest that kind of effort, they demand a proof from us.

The Thin Set Theorem - as an object of study in the foundations of mathematics - first appeared publicly in [Fr00], and in print in [FS00], p. 139. There we remark that it trivially follows from the following well known Free Set Theorem for N.

FREE SET THEOREM. Let $k \geq 1$ and $f:N^k \rightarrow N$. There exists infinite $A \subseteq N$ such that for all $x \in A^k$, $f(x_1,...,x_k) \in A \rightarrow f(x_1,...,x_k) \in \{x_1,...,x_k\}$.

The implication is merely the observation that if A obeys the conclusion of the Free Set Theorem, then $A\setminus\{\min(A)\}$ obeys the conclusion of the Thin Set Theorem ($\min(A)$ is not a value of f on $(A\setminus\{\min(A)\})^k$).

The Free Set Theorem is easily obtained from the infinite Ramsey theorem in a well known way. Choose infinite $A \subseteq N$ such that the truth value of $f(x_1,...,x_k) = y$ depends only on the order type of $x_1,...,x_k,y$, provided $x_1,...,x_k,y \in A$. If $f(x_1,...,x_k) = y \notin \{x_1,...,x_k\}$, where $x_1,...,x_k,y \in A$, then we can move $x_1,...,x_k,y$ around in A so that we have

$$f(x_1,...,x_k) = y$$
$$f(x_1,...,x_k) = y'$$

where y' is the element of A right after y. This is a contradiction.

In [FS00], p. 139-140, we presented our proof that the Thin Set Theorem is not provable in ACA_0. A proof of our result that the Thin Set Theorem for binary functions cannot be proved in WKL_0 appears in [CGHJ05]. [CGHJ05] also contains an exposition of our proof that the Thin Set Theorem is not provable in ACA_0. It is easy to see that the Free Set Theorem, and hence the Thin Set Theorem, for arity 1, is provable in RCA_0.

The metamathematical status of the Thin Set Theorem and the Free Set Theorem are not known.

This is in sharp contrast to the well known status of the infinite Ramsey theorem. The infinite Ramsey theorem for any fixed exponent $n \geq 3$ is provably equivalent to ACA_0 over RCA_0. The infinite Ramsey theorem stated for all exponents is provably equivalent to a system ACA' over RCA_0, defined as follows.

DEFINITION 1.4.1. The system ACA′ is the system ACA_0 together with $(\forall n)(\forall x \subseteq \omega)$(the n-th Turing jump of x exists). This is logically equivalent to RCA_0 + $(\forall n)(\forall x \subseteq \omega)$(the n-th Turing jump of x exists).

It is well known that ACA′ is a fragment of the system ACA, which is ACA_0 together with full induction. See [Si99] for a discussion of ACA_0 and other subsystems of second order arithmetic, including RCA_0 (used throughout this book).

We originally introduced ACA′ around the time we set up Reverse Mathematics (but after we introduced $RCA_0,ACA_0,WKL_0,ATR_0,\Pi^1_1\text{-}CA_0$), in order to analyze the usual infinite Ramsey theorem. We straightforwardly adapted part of the recursion theoretic treatment due to Carl Jockusch of Ramsey theorem to show that RT is provably equivalent to ACA′ over RCA_0.

It must be mentioned that the metamathematical status of the infinite Ramsey theorem for exponent 2 is not known, although there has been considerable progress on this. See [CJS01].

The main metamathematical open questions concerning the Thin Set Theorem and the Free Set Theorem are these. Do they imply ACA_0 over RCA_0 for fixed exponents? Do they imply Ramsey's theorem (when stated for arbitrary exponents)? By the previous remarks, these questions are equivalent to the following. For fixed exponents, are they equivalent to ACA_0 over RCA_0? Are they equivalent to ACA′ over RCA_0 (or over ACA_0)? It is possible that the Free Set Theorem is stronger than the Thin Set Theorem over RCA_0.

We now present a proof of our Thin Set Theorem privately communicated to us by Jeff Remmel, that does not pass through the Free Set Theorem.

DEFINITION 1.4.2. Let $f:N^k \rightarrow N$. We define ot(k) to be the number of order types of k tuples from N.

We now define a coloring of the k-tuples x from N according to the value f(x). Specifically, the color f(x) is assigned to x if $f(x) \in [1,ot(k)]$, and the color 0 is assigned to x otherwise.

By the usual infinite Ramsey theorem for k tuples, we obtain $A \in INF$ such that for all $m \in [0,ot(k)]$,

$(\forall x,y \in A^k)$(if x,y have the same order type then $f(x) = m \leftrightarrow f(y) = m$).

It is clear that the $x \in A^k$ of any given order type can only map to at most one element of $[0,ot(k)]$. Hence $rng(f|A^k) \cap [0,ot(k)]$ has at most ot(k) elements. Therefore $rng(f|A^k)$ omits at least one element of $[0,ot(k)]$.

From this proof, we can conclude the following strong form of the Thin Set Theorem.

THIN SET THEOREM ([0,ot(k)]). For all $f:N^k \rightarrow [0,ot(k)]$ there exists infinite $A \subseteq N$ such that $fA \neq [0,ot(k)]$.

The function ot(k) has been well studied in the literature. Let ot(k) be the number of order types of elements of N^k. It is obvious that $ot(k) \leq k^k$ (every element on N^k has the same order type as an element of $[k]^k$), and a straightforward inductive argument shows that $ot(k) \leq 2k(k!)$. In [Gr62] it is shown that ot(k) is asymptotic to $(k!/2) \ln^{k+1} 2$.

The metamathematical status of this form of the Thin Set Theorem can be easily determined as follows.

THEOREM 1.4.1. Thin Set Theorem ([0,ot(k)]), for fixed exponents, or for exponent 3, is provably equivalent to ACA_0 over RCA_0. Thin Set Theorem ([0,ot(k)]) is provably equivalent to ACA′ over RCA_0. RCA_0 refutes Thin Set Theorem ([1,ot(k)]) in every exponent k.

Proof: Evidently, Thin Set Theorem ([0,ot(k)]) for fixed exponent k is provable in ACA_0. For general exponents k (as a free variable), it is provable from the infinite Ramsey theorem, and therefore in ACA′.

We now argue in RCA_0. Let $k \geq 2$ (as a free variable). Assume the Thin Set theorem ([0,ot(k)]). We derive the infinite Ramsey theorem for exponent k and $ot(k) \geq 2$ colors, as follows. Let $f:[N]^k \rightarrow \{0,1\}$, where $[A]^k$ is the set of all subsets of A of cardinality k. Let $\alpha_1,\ldots,\alpha_{ot(k)}$ be an enumeration of all of the order types of k-tuples from N, where α_1 is the order type of $1,2,\ldots,k$. We now define $g:N^k \rightarrow [0,ot(k)]$ as follows.

case 1. $x_1 < \ldots < x_k$. Set $g(x_1,\ldots,x_k) = f(\{x_1,\ldots,x_k\})$.

case 2. Otherwise. Let the order type of $x_1,\ldots,x_k$ be α_i, and set $g(x_1,\ldots,x_k) = i \geq 2$.

Let $A \subseteq N$ be infinite, where gA does not contain [0,ot(k)]. Clearly gA contains [2,ot(k)]. Since gA does not contain both 0,1, we see that f is constant on $[A]^k$.

If we fix k = 3 then we obtain the infinite Ramsey theorem for exponent 3 with 2 colors, and hence ACA_0, over RCA_0.

If we use k as a free variable, then we obtain the infinite Ramsey theorem, and hence ACA', over RCA_0.

For the final claim, use $f:N^k \to [1,ot(k)]$ by $f(x_1,\ldots,x_k) = i$, where the order type of $(x_1,\ldots,x_k)$ is the i-th order type of elements of N^k. QED

What if we use another, simpler, function of k?

THIN SET THEOREM ($[1,k^k]$). For all $f:N^k \to [1,k^k]$ there exists infinite $A \subseteq N$ such that $fA \neq [0,k^k]$.

We do not know the status of THIN SET THEOREM ($[1,k^k]$). It obviously follows from THIN SET THEOREM ([1,ot(k)]), exponent by exponent.

From the point of view of the partition calculus, it is more natural to use $[N]^k$ instead of N^k. Here $[N]^k$ is the set of all subsets of N of cardinality k. The square bracket partition relation

$$\omega \to [\omega]^k_\omega$$

can be stated as follows. Let $[A]^k$ be the set of all subsets of A of cardinality k.

THIN SET THEOREM (unordered tuples). For all $f:[N]^k \to N$ there exists infinite $A \subseteq N$ such that $f[[A]^k] \neq N$.

THEOREM 1.4.2. The Thin Set Theorem and the Thin Set Theorem (unordered tuples) are provably equivalent in RCA_0.

Proof: The forward direction is obvious. Now assume the Thin Set Theorem (unordered tuples). Let $f:N^k \to N$, $k \geq 1$. We will prove the existence of an infinite $A \subseteq N$ such that $fA \neq N$.

We can assume that Ramsey's theorem for two colors fails (arbitrary exponents), since otherwise we conclude the Thin Set Theorem. Let $g:[N]^r \to \{0,1\}$ be a counterexample to Ramsey's theorem. We construct a function $h:[N]^{k+r\cdot ot(k)} \to N$ such that for all infinite $A \subseteq N$, $fA \subseteq h[[A]^{k+r\cdot ot(k)}]$. So if $h[[A]^{k+r\cdot ot(k)}] \neq N$ then $fA \neq N$.

Define $h:[N]^{k+r\cdot ot(k)} \to N$ as follows. Let $x_1 < \ldots x_k < y_1 < \ldots y_{r\cdot ot(k)}$ be given. Apply g to the ot(k) successive blocks of length r in $y_1,\ldots,y_{r\cdot ot(k)}$ to obtain ot(k) bits. Let t be the position of the first one of these bits that is 0. If they are all 1's, then set t = 1. Clearly $1 \leq t \leq ot(k)$. Let α be the t-th order type of length k, in some prearranged listing of order types of elements of N^k. Let $y \in \{x_1,\ldots,x_k\}^k$ be minimum (have minimum max) of order type α. Set $h(\{x_1,\ldots,x_k,y_1,\ldots,y_{r\cdot ot(k)}\}) = f(y)$. By the Thin Set Theorem (unordered tuples), let $A \in INF$ be such that $h[[A]^{k+r\cdot ot(k)}] \neq N$. Since g is a counterexample to Ramsey's theorem, all ot(k) length bit sequences appear in the construction of hA. Hence all length k order types are used. Therefore $fA \subseteq h[[A]^{k+r\cdot ot(k)}] \neq N$. QED

We now discuss some other kinds of strengthenings of the Thin Set Theorem. Let $W \subseteq POW(N)$.

THIN SET PROPERTY FOR W. For all $f \in MF$ there exists $A \in W$ such that $fA \neq N$.

DEFINITION 1.4.3. The upper density of $A \subseteq N$ is given by

$$\limsup_{n\to\infty}|A \cap [0,n)|/n.$$

DEFINITION 1.4.4. The upper logarithmic density of $A \subseteq N$ is given by

$$\limsup_{n\to\infty}\log|A \cap [0,n)|/n.$$

THEOREM 1.4.3. The Thin Set Property fails for positive upper density. In fact, it fails for arity 3.

Proof: Let $f:N^3 \to N$ be defined by

$$f(a,b,c) = (c-a)/(b-a) - 2 \text{ if this lies in } N; \quad 0 \text{ otherwise.}$$

Let $A \subseteq N$ have positive upper density. Szemeredi's theorem, [Gow01], asserts that every set of positive upper density contains arbitrarily long arithmetic progressions. Let $p \geq 2$ and $a,a+b,\ldots,a+pb$ be an arithmetic progression in A of length p+1, where $a \geq 0$ and $b \geq 1$. Then $f(a,a+b,a+pb) = (pb)/b - 2 = p-2$. Hence fA = N. QED

We do not know if Theorem 1.4.3 can be improved to arity 2, and we do not know if the Thin Set Property holds for upper logarithmic density. We do not know any interesting necessary or sufficient conditions on W so that the Thin Set Property holds for W.

DEFINITION 1.4.5. Let κ be an infinite cardinal. The Thin Set Property for κ asserts the following. For all $f:\kappa^n \to \kappa$, there exists $A \subseteq \kappa$ of cardinality κ such that $fA \neq \kappa$.

THEOREM 1.4.4. [To87], Theorem 5.2. The Thin Set Property fails for the successor of any regular cardinal. In fact, it fails even for unordered 2-tuples. In particular, it fails for unordered 2-tuples in the case of ω_1.

The Thin Set Property is well known to hold of all weakly compact cardinals, since it follows from $\kappa \to \kappa^n{}_m$. In fact, the Thin Set Property holds for weakly compact cardinals in the strong [0,ot(k)] form. For more on the Thin Set Property, see [BM90], Theorem 4.12, and [EHMR84], Theorem 54.1. Also see [Sh95].

The Thin Set Theorem makes perfectly good sense in any BRT setting (V,K). It simply asserts that for all $f \in V$, there exists $A \in K$, such that $fA \neq U$. Here U is the universal set associated with the BRT setting (V,K), as defined in section 1.1.

We now explore the Thin Set Theorem on some BRT settings in real analysis. There have no intention of exhausting anything like a fully representative sample of all interesting BRT settings in real analysis. We only present a very limited sample.

We will see that the Thin Set Theorem, which is the simplest nontrivial statement in all of BRT, depends very much on the choice of BRT setting. We expect that the same is true for a huge variety of statements in BRT, in a rather deep way.

We first consider only unary functions from $\Re$ to $\Re$. It is natural to extend the investigation to incorporate families of functions whose domains are of various kinds (open, semialgebraic, etc.). This is beyond the scope of this book.

We now restrict attention to 8 families of functions from $\Re$ into $\Re$, and 9 families of subsets of $\Re$.

FCN($\Re$,$\Re$). All functions from $\Re$ to $\Re$.
BFCN($\Re$,$\Re$). All Borel functions from $\Re$ to $\Re$.
CFCN($\Re$,$\Re$). All continuous functions from $\Re$ to $\Re$.
C^1FCN($\Re$,$\Re$). All C^1 functions from $\Re$ to $\Re$.
C^∞FCN($\Re$,$\Re$). All C^∞ functions from $\Re$ to $\Re$.
RAFCN($\Re$,$\Re$). All real analytic functions from $\Re$ to $\Re$.
SAFCN($\Re$,$\Re$). All semialgebraic functions from $\Re$ to $\Re$.
CSAFCN($\Re$,$\Re$). All continuous semialgebraic functions from $\Re$ to $\Re$.

cSUB($\Re$). All subsets of $\Re$ of cardinality c.
UNCLSUB($\Re$). All uncountable closed subsets of $\Re$.
NOPSUB($\Re$). All nonempty open subsets of $\Re$.
UNOPSUB($\Re$). All unbounded open subsets of $\Re$.
DEOPSUB($\Re$). All open dense subsets of $\Re$.
FMOPESUB($\Re$). All open subsets of $\Re$ of full measure.
CCOPSUB($\Re$). All open subsets of $\Re$ whose complement is countable.
FCSUB($\Re$). All subsets of $\Re$ whose complement is finite.
≤1CSUB($\Re$). All subsets of $\Re$ whose complement has at most one element.

These two lists alone provide 8•9 = 72 BRT settings. We conjecture that there are substantial BRT differences between these 72, except that perhaps C^1FCN($\Re$,$\Re$) and C^∞FCN($\Re$,$\Re$) have the same BRT behavior. We won't venture an opinion on that.

Note that here we only focus on just one statement of IBRT in one function and one set: the Thin Set Theorem.

THEOREM 1.4.5. The Thin Set Theorem holds on (FCN($\Re$,$\Re$),cSUB($\Re$)), (BFCN($\Re$,$\Re$),UNCLSUB($\Re$)), (CFCN($\Re$,$\Re$),FMOPSUB($\Re$)), (C^1FCN($\Re$,$\Re$),CCOPSUB($\Re$)), (SAFCN($\Re$,$\Re$),FCSUB($\Re$)), (CSAFCN($\Re$,$\Re$),≤1CSUB($\Re$)).

Proof: Let $f:\Re \to \Re$. We can assume that rng(f) = $\Re$. Let A = $f^{-1}[0,1]$. Then A has cardinality c and fA = [0,1] ≠ $\Re$.

Let $f:\mathfrak{R} \to \mathfrak{R}$ be Borel. According to [Ke94], exercise 19.8, there exists a nowhere dense perfect set $P \subseteq \mathfrak{R}$ such that f is either 1-1 continuous on P, or f is constant on P. In either case, fP is nowhere dense, and so $fP \neq \mathfrak{R}$. See Theorem 1.4.7 for the sharper multivariate form of this result.

Let $f:\mathfrak{R} \to \mathfrak{R}$ be continuous. Then there exists $x \in \mathfrak{R}$ such that $f^{-1}(x)$ is a closed set of measure 0. Let $A = \mathfrak{R}\backslash f^{-1}(x)$. Then A is an open subset of $\mathfrak{R}$ of full measure, and $fA \neq \mathfrak{R}$.

Let $f:\mathfrak{R} \to \mathfrak{R}$ be C^1. By Sard's theorem, $S = \{f(x): f'(x) = 0\}$ has measure zero. Suppose $f^{-1}(x)$ is uncountable. Then $f^{-1}(x)$ contains a limit u, where obviously $f'(u) = 0$. Hence $f(u) \in S$, and so $x \in S$. Thus $\{x: f^{-1}(x) \text{ is uncountable}\} \subseteq S$. Since S has measure zero, let x be such that $f^{-1}(x)$ is countable. Then $\mathfrak{R}\backslash f^{-1}(x)$ is an open subset A of $\mathfrak{R}$ whose complement is countable, and $fA \neq \mathfrak{R}$. (The use of Sard's theorem here was suggested to the author by Gerald Edgar.)

Let $f:\mathfrak{R} \to \mathfrak{R}$ be semialgebraic. Every $f^{-1}(x)$ is finite or contains a nonempty open interval. Hence some $f^{-1}(x)$ is finite. Let $A = \mathfrak{R}\backslash f^{-1}(x)$. Then A is a subset of $\mathfrak{R}$ whose complement is finite, and $fA \neq \mathfrak{R}$.

Let $f:\mathfrak{R} \to \mathfrak{R}$ be continuous and semialgebraic. We can assume that $\text{rng}(f) = \mathfrak{R}$. Hence there exists $x > 0$ such that

$f:[x,\infty) \to [f(x),\infty)$ is strictly increasing and onto, and
$f:(-\infty,-x] \to (-\infty,f(-x)]$ is strictly increasing and onto; or

$f:[x,\infty) \to (-\infty,f(x)]$ is strictly decreasing and onto, and
$f:(-\infty,-x] \to [f(-x),\infty)$ is strictly decreasing and onto.

This can be proved using the well known structural properties of semialgebraic $f:\mathfrak{R} \to \mathfrak{R}$ as treated in [Dr98], Chapter 1.
In the first case, choose $y > \max(f[(-\infty,x]])$, so that the value y is attained only on $[x,\infty)$, in which case $f^{-1}(y)$ has exactly one element. In the second case, choose $y < \min(f[(-\infty,x]])$, so that the value y is also attained only on $[x,\infty)$, in which case $f^{-1}(y)$ has exactly one element. In both cases, we can find y by the continuity of f. Let $A = \mathfrak{R}\backslash f^{-1}(y)$. Then A is a subset of $\mathfrak{R}$ whose complement has cardinality 1, where $fA \neq \mathfrak{R}$. QED

THEOREM 1.4.6. The Thin Set Theorem fails on $(\mathrm{FCN}(\Re,\Re),\mathrm{UNCLSUB}(\Re))$, $(\mathrm{BFCN}(\Re,\Re),\mathrm{NOPSUB}(\Re))$, $(\mathrm{CFCN}(\Re,\Re),\mathrm{CCOPSUB}(\Re))$, $(\mathrm{RAFCN}(\Re,\Re),\mathrm{FCSUB}(\Re))$, $(\mathrm{SAFCN}(\Re,\Re),\leq 1\mathrm{CSUB}(\Re))$.

Proof: We first construct a function α which maps every uncountable closed subset of $\Re$ to a subset of cardinality c, such that $A \neq B \rightarrow \alpha(A) \cap \alpha(B) = \emptyset$. This is done by a transfinite construction of length c. We use an enumeration of the uncountable closed subsets of $\Re$ of length c.

At any stage, we have a function β which maps every uncountable closed subsets of $\Re$ to a subset of cardinality < c, where < c uncountable closed subsets are assigned a nonempty subset. At the next stage, we can add one more point to each of the nonempty subsets thus far, and assign a subset of cardinality 1 to the first uncountable closed subset of $\Re$ that was previously assigned $\emptyset$. Since uncountable closed subsets of $\Re$ are of cardinality c, there is no problem making sure that the sets assigned are pairwise disjoint.

For each uncountable closed A, map $\alpha(A) \subseteq A$ onto $\Re$. By the disjointness of the $\alpha(A)$, we can take the union of these functions, and then extend this union arbitrary to $f:\Re \rightarrow \Re$. Clearly for all uncountable closed A, we have $fA = \Re$. This establishes the first claim.

It is well known that there exists a continuous $f:\Re \rightarrow \Re$ such that each $f^{-1}(x)$ is uncountable. E.g., start with a continuous $g:K \rightarrow K$ such that each $f^{-1}(x)$, $x \in K$, is uncountable, where K is the Cantor set. (Take g(x) to be the real number whose base 3 expansion is digits number 1,3,5,7,... in the base 3 expansion of x). Compose with a continuous map from K onto [0,1] to obtain a continuous $h:K \rightarrow [0,1]$ such that each $h^{-1}(x)$, $x \in [0,1]$, is uncountable. Then extend to a continuous $J:[0,1] \rightarrow [0,1]$ with this property. Then paste copies of the functions J+n, $n \in Z$, together with some filler, to obtain $f:\Re \rightarrow \Re$ such that each $f^{-1}(x)$ is uncountable. Let $A \subseteq \Re$ have countable complement. Then A must meet each $f^{-1}(x)$. Hence $fA = \Re$.

It is obvious that there exists real analytic $f:\Re \rightarrow \Re$ such that each $f^{-1}(x)$ is infinite - e.g., xsin(x). Let $A \subseteq \Re$ have finite complement. Then A must meet each $f^{-1}(x)$. Hence $fA = \Re$.

It is obvious that there are semialgebraic $f:\Re \to \Re$ such that each $f^{-1}(x)$ has at least 2 elements. Let $A \subseteq \Re$ omit at most one real number. Then A must meet each $f^{-1}(x)$. Hence fA $= \Re$.

It remains to treat (BFCN($\Re$,$\Re$),NOPSUB($\Re$)). We define $f:[0,1] \to [0,1]$ and $g:[0,1] \to Z$ as follows. Let $x \in [0,1]$ and let $b_1,b_2,\ldots$ be the base 2 expansion of x. Let x* be greatest $m \geq 1$ such that $b_1,b_2,\ldots,b_m =$ b_8^m+1,b_8^m+3,...,b_8^m+2m-1. If m does not exist, set f(x) = 0. If m exists, set f(x) to be the real number $b_{4m},b_{4m+4},b_{4m+8},\ldots$. Set $|g(x)|$ to be the least $i \geq 0$ such that $b_{4m+4i+6} = 0$ if it exists; 0 otherwise. Choose the sign of g(x) according to the bit b_{4m+2}.

We claim that for all nonempty open $A \subseteq [0,1]$ and $y \in [0,1]$ and $k \in Z$, there exists $x \in A$ such that f(x) = y and g(x) = k.

To see this, let A,y,k be as given. Let $(p,q) \subseteq A$ be nondegenerate, where p,q are dyadic rationals, $0 \leq p < q \leq 1$. Let $b_1,\ldots,b_n,0,0,0,\ldots$ be the base 2 expansion of p and $c_1,\ldots,c_n,0,0,0,\ldots$ be the base 2 expansion of q. By continuing the base 2 expansion $b_1,\ldots,b_n$, we can arrange that x* = m exists, m > n, without committing ourselves to any of the base 2 digits beyond b_n in even positions. We then arrange $b_{4m},b_{4m+4},b_{4m+8},\ldots$ to be a binary expansion of y. We have thus arranged f(x) = y. We can also arrange g(x) = k.

Finally, we define $h:\Re \to \Re$ by $h(x) = f(x-\lfloor x\rfloor)+g(x-\lfloor x\rfloor)$. Here $\lfloor x\rfloor$ is the floor of x, which is the greatest integer $\leq$ x.

Now let $A \subseteq \Re$ be a nonempty open subset of $\Re$ and $y \in \Re$. Let (p,q) + k be an open interval contained in A, where $0 < p < q < 1$. We can find $x \in (p,q)$ such that $f(x) = y-\lfloor y\rfloor$ and $g(x) = \lfloor y\rfloor$. Then $h(x+\lfloor y\rfloor) = f(x)+g(x) = y$ as required. QED

Note that in the above development, we have not come across a distinction between C^1FCN($\Re$,$\Re$), C^∞FCN($\Re$,$\Re$), and RAFCN($\Re$,$\Re$). We suspect that important distinctions will arise as we go deeper into BRT.

We now consider the corresponding 8 families of multivariate functions from $\Re$ to $\Re$. I.e., functions whose domain is some $\Re^n$ and whose range is a subset of $\Re$. We use the same 9 families of subsets of $\Re$.

$FCN(\Re^*,\Re)$. All multivariate functions from $\Re$ to $\Re$.
$BFCN(\Re^*,\Re)$. All multivariate Borel functions from $\Re$ to $\Re$.
$CFCN(\Re^*,\Re)$. All multivariate continuous functions from $\Re$ to $\Re$.
$C^1FCN(\Re^*,\Re)$. All multivariate C^1 functions from $\Re$ to $\Re$.
$C^\infty FCN(\Re^*,\Re)$. All multivariate C^∞ functions from $\Re$ to $\Re$.
$RAFCN(\Re^*,\Re)$. All multivariate real analytic functions from $\Re$ to $\Re$.
$SAFCN(\Re^*,\Re)$. All multivariate semialgebraic functions from $\Re$ to $\Re$.
$CSAFCN(\Re^*,\Re)$. All multivariate continuous semialgebraic functions from $\Re$ to $\Re$.

Here the domains of all functions considered are Cartesian powers of $\Re$, and the ranges are all subsets of $\Re$.

THEOREM 1.4.7. The Thin Set Theorem holds on $(BFCN(\Re^*,\Re),UNCLSUB(\Re))$, $(CFCN(\Re^*,\Re),NOPSUB(\Re))$. If c is a real valued measurable cardinal then the Thin Set Theorem holds on $(FCN(\Re^*,\Re),cSUB(\Re))$. If κ is a weakly compact cardinal internal to a countable transitive model M of ZFC, and we force over M with finite partial functions from κ into $\{0,1\}$ under inclusion, then the Thin Set Theorem holds on $(FCN(\Re^*,\Re),cSUB(\Re))$ in the generic extension.

Proof: We start with the first claim. It is convenient to prove a somewhat stronger result: that the Thin Set Theorem holds on $BFCN(\Re^*,\Re)$ with the uncountable closed subsets of $\Re$ that are unbounded.

We will rely on the well known adaptation of forcing technology for such applications. Let K be the usual Cantor space $\{0,1\}^N$. We first show the following.

#) Let $\alpha_0,\alpha_1,\ldots$ be Borel functions from K^n into K. There exists a perfect $K' \subseteq K$ such that each $\alpha_i[K'^{n\neq}]$ is nowhere dense in K, where $K'^{n\neq}$ is the set of all n-tuples of distinct elements of K'.

We fix a countable admissible set X containing a sequence of codes for the Borel functions $\alpha_0,\alpha_1,\ldots$. We will freely use forcing over X.

We write $\{0,1\}^{<N} = \cup\{\{0,1\}^i: i \geq 0\}$. Here $\{0,1\}^i$ is the set of functions from i into $\{0,1\}$, where $i = \{0,\ldots,i-1\}$.

We use $(f_1,\ldots,f_n)$ as the name of an undetermined generic element of K^n. The statements we will force are of the form

$$\alpha_i(f_1,\ldots,f_n)(k) = j,$$

where $i,k \geq 0$ and $j \in \{0,1\}$. Forcing will be defined as usual over X for conditions $p = (x_1,\ldots,x_n) \in (\{0,1\}^{<N})^n$. The notion of generic $(g_1,\ldots,g_n) \in K^n$ is defined as usual using dense sets of conditions lying in X.

A 1-condition is an element of $\{0,1\}^{<N}$. The conditions are of course n-tuples of 1-conditions.

We will now build a perfect finite sequence tree T of conditions. The root is the empty sequence of conditions. The vertices will have the form $\langle p_1,\ldots,p_b\rangle$, $p \geq 0$, where $p_1 \subseteq \ldots \subseteq p_b$ are 1-conditions. Here b is the length of the vertex.

At every level i in T, the 2^i vertices will all have a common structure in that they will all have the form $\langle p_1,\ldots,p_i\rangle$, and any two of them, $\langle p_1,\ldots,p_i\rangle \neq \langle p_1',\ldots,p_i'\rangle$, will have

$$\begin{gathered} \mathrm{lth}(p_1) = \mathrm{lth}(p_1'). \\ \ldots \\ \mathrm{lth}(p_i) = \mathrm{lth}(p_i'). \\ p_i \neq p_i'. \end{gathered}$$

Here $p_1,\ldots,p_i$ are 1-conditions with $p_1 \subseteq \ldots \subseteq p_i$.

Let $D_1,D_2,\ldots$ be an enumeration of the dense sets of conditions lying in X, which are closed upward. Let $w_1,\ldots,w_n$ be the last terms of n distinct vertices at level i of T. These w's are 1-conditions. We require that the condition $(w_1,\ldots,w_n)$ lie in $D_1 \cap \ldots \cap D_i$. This will guarantee that any sequence of n distinct infinite paths through T will be a generic element of K^n (when flattened out in the obvious way).

Suppose we have constructed the i-th level of T, $i \geq 0$. We now show how to construct the i+1-st level of T. Let $w_1,\ldots,w_{2^\wedge i}$ be the last terms of the vertices of T at the i-th level. Recall that the w's are distinct 1-conditions of the same lengths.

Let $i' >> i$. Before doing any splitting, first extend the w's to $w_1^*,\ldots,w_{2^i}^*$ so that every sequence of distinct elements $u_1,\ldots,u_n$ of $\{w_1^*,\ldots,w_{2^i}^*\}$ decides all forcing statements

$$\alpha_s(f_1,\ldots,f_n)(t) = 0,\ 0 \leq s,t < i'.$$

This will ensure that for all $s \geq 1$, the number of possible first i' bits of values of α_s, at the n-tuples of distinct infinite paths through T, will be smaller than, say, the square root of $2^{i'}$. This will guarantee the desired nowhere density of the set of all values of α_s, at the n-tuples of distinct infinite paths through T.

There is no problem meeting the earlier requirements by further extensions and by 2 splitting. This establishes #). We now sharpen #).

##) Let $\alpha_0,\alpha_1,\ldots$ be Borel functions from K^n into K. There exists a perfect $K' \subseteq K$ such that each $\alpha_i K'$ is nowhere dense in K. In particular, $\cup\{\alpha_i K': i \geq 0\} \neq K$.

Let the α's be as given. For each $i \geq 0$ and $1 \leq j_1,\ldots,j_n \leq n$, put the function

$$\beta(x_1,\ldots,x_n) = \alpha_i(x_{j1},\ldots,x_{jn})$$

in a new list, and apply #) to these β's. Let K' be given by #). Then each $\alpha_i K'$ is a finite union of the various $\beta[K'^{n\neq}]$, and is therefore nowhere dense in K.

The final part of ##) is by the Baire category theorem for K.

Now let $f:\Re^n \to \Re$ be Borel. We now think of $K \subseteq \Re$ by viewing each element of K as a base 3 expansion with only 0's and 2's. Let $h:\Re \to K$ be a Borel bijection.

For all $\delta \in N^n$, let $g_\delta:K^n \to K$ be defined by

$$g_\delta(x) = h(f(x+\delta)).$$

By ##), let $K' \subseteq K$ be a perfect set such that $\cup\{g_\delta K': \delta \in N^n\} \neq K$. We claim that

$$A = \cup\{K'+i: i \geq 0\} \subseteq \Re$$

is uncountable, closed, and unbounded.

$$fA \neq \Re.$$

Since $K' \subseteq [0,1]$ is uncountable and closed, clearly, A is uncountable, closed, and unbounded.

Let $u \in K \backslash \cup\{g_\delta K': \delta \in N^n\}$. Then for all $\delta \in N^n$, $u \notin g_\delta K'$. Hence for all $\delta \in N^n$ and $x \in K'$, $u \neq g_\delta x = h(f(x+\delta))$. Clearly, for all $\delta \in N^n$ and $x \in K'$, $h^{-1}u \neq f(x+\delta)$. Hence for all $i \geq 0$, $h^{-1}u \notin f(K+i)$. Hence $h^{-1}u \notin fA$.

For the second claim of the Theorem, let $f:\Re^k \to \Re$ be continuous. Then $f([0,1]) = f([0,1]^k)$ is compact, and therefore not $\Re$.

The real valued measurable claim is proved in [So71], Lemma 14, page 406.

For the final claim, let M be a countable transitive model of ZFC with the internal weakly compact cardinal κ. We force to create a generic set $a \subseteq \kappa$ using finite conditions. It is convenient to write this generic set as mutually generic subsets of ω, $\{a_\alpha \subseteq \omega: \alpha < \kappa\}$.

Fix a generic extension $M^* = M[\{a_\alpha \subseteq \omega: \alpha < \kappa\}]$ using this notion of forcing. It suffices to show that in M^*, for every $f:\wp(\omega)^n \to \wp(\omega)$ there exists $A \subseteq \wp(\omega)$ of cardinality κ such that $fA \neq \wp(\omega)$.

Let τ be a forcing term representing f in M^*. Let p force $\tau:\wp(\omega)^n \to \wp(\omega)$, where the condition p is compatible with the generic object.

Since κ is a weakly compact cardinal in M, κ is strongly inaccessible, and $\kappa \to \kappa^n{}_\omega$ in M. Let σ be one of the finitely many possible order types of tuples $(\alpha_1,\ldots,\alpha_n,\gamma)$ with $\gamma \neq \alpha_1,\ldots,\alpha_n$. Let $E \in M$, $E \subseteq \kappa$, E unbounded, be such that

for all $\alpha_1,\ldots,\alpha_n,\gamma < \kappa$, $(\alpha_1,\ldots,\alpha_n,\gamma)$ of type σ, $\tau(a_{\alpha 1},\ldots,a_{\alpha n}) = a_\gamma$ is not forced by any extension of p; or

for all $\alpha_1,\ldots,\alpha_n$ $\gamma < \kappa$, $(\alpha_1,\ldots,\alpha_n,\gamma)$ of type σ, $\tau(a_{\alpha 1},\ldots,a_{\alpha n}) = a_\gamma$ is forced by some extension of p.

Suppose that the latter holds. Let $\alpha_1,\ldots,\alpha_n,\gamma$ be of type σ, where there are uncountably many γ' such that $(\alpha_1,\ldots,\alpha_n,\gamma')$ has type σ. Then the corresponding extensions of p must be incompatible. This violates the fact that this notion of forcing has the countable chain condition in M.

Hence we have

$$\text{for all } \alpha_1,\ldots,\alpha_n,\gamma < \kappa,\ (\alpha_1,\ldots,\alpha_n,\gamma) \text{ of type } \sigma,$$
$$\tau(a_{\alpha 1},\ldots,a_{\alpha n}) = a_\gamma \text{ is not forced by any extension of p}$$

assuming that σ is an order type with last term different than all earlier terms. Hence

$$\text{for all } \alpha_1,\ldots,\alpha_n,\gamma < \kappa, \text{ if } \gamma \neq \alpha_1,\ldots,\alpha_n \text{ then } \tau(a_{\alpha\_1},\ldots,a_{\alpha\_n})$$
$$= a_\gamma \text{ is not forced by any extension of p.}$$

In M*,

$$\text{for all } \alpha_1,\ldots,\alpha_n < \kappa,\ f(a_{\alpha\_1},\ldots,a_{\alpha\_n}) \neq a_{\alpha\_1},\ldots,a_{\alpha\_n} \rightarrow$$
$$(\forall\gamma \neq \alpha_1,\ldots,\alpha_n)(f(a_{\alpha\_1},\ldots,a_{\alpha\_n}) \neq a_\gamma).$$

Let $A = \{a_\alpha\colon \alpha \in E\}$. Then in M*, $|A| = \kappa$, and

$$\text{for all } x_1,\ldots,x_n < \kappa,\ f(x_1,\ldots x_n) \neq x_1,\ldots,x_n \rightarrow$$
$$f(x_1,\ldots,x_n) \notin A.$$

Clearly $\min(A) \notin f(A\backslash\{\min(A)\})$. QED

THEOREM 1.4.8. The Thin Set Theorem fails on $(SAFCN(\Re^3,\Re),NOPSUB(\Re))$, $(CSAFCN(\Re^3,\Re),UNOPSUB(\Re))$. If the continuum hypothesis holds then the Thin Set Theorem fails on $(FCN(\Re^2,\Re),cSUB(\Re))$.

Proof: Let $f:\Re^3 \rightarrow \Re$ be given by $f(x,y,z) = 1/(x-y) + 1/(x-z)$ if defined; 0 otherwise. Then f is semialgebraic. Let $a,b \in \Re$, $a < b$. We claim that $f[(a,b)] = \Re$. To see this, let $u \in \Re$. Fix $x \in (a,b)$. We can find $y,z \in (a,b)$ such that $1/(x-y)$ and $1/(x-z)$ are any two reals with sufficiently large absolute values. Hence we can find $y,z \in (a,b)$ such that $f(x,y,z) = u$.

Let $f:\Re^3 \rightarrow \Re$ be given by $x(y-z)$. Then f is continuous and semialgebraic. Let A be an unbounded open subset of $\Re$. We claim that $fS = \Re$. To see this, let $u \in \Re$. Let $(a,b) \subseteq A$, where $a < b$. Choose $z \in A$ such that $|z| > |u/(b-a)|$. Then $|u/z| < b-a$. Let $x,y \in (a,b)$, where $x-y = u/z$. Then $f(x,y,z) = u$.

The final claim is by Theorem 1.4.4. QED

Note that the counterexamples above are in 3 dimensions.

THEOREM 1.4.9. The Thin Set Theorem holds on $(\mathrm{SAFCN}(\mathfrak{R}^2,\mathfrak{R}),\mathrm{UNOPSUB}(\mathfrak{R}))$. The Thin Set Theorem fails on $(\mathrm{CSAFCN}(\mathfrak{R}^2,\mathfrak{R}),\mathrm{DEOPSUB}(\mathfrak{R}))$ and $(\mathrm{RAFCN}(\mathfrak{R}^2,\mathfrak{R}),\mathrm{UNOPSUB}(\mathfrak{R}))$.

Proof: Let $E \subseteq \mathfrak{R}^2$ be semialgebraic. We say that $A \subseteq \mathfrak{R}^2$ is small if and only if $(\forall x >> 0)(\forall y >> x)((x,y) \notin A)$. We claim that for any disjoint semialgebraic $A,B \subseteq \mathfrak{R}^2$, A is small or B is small.

To see this, let $A,B \subseteq \mathfrak{R}^2$ be pairwise disjoint semiaglebraic sets, where A,B are not small. Then

$$\neg(\forall x >> 0)(\forall y >> x)((x,y) \notin A).$$
$$\neg(\forall x >> 0)(\forall y >> x)((x,y) \notin B).$$

By the o-minimality of the field of real numbers,

$$(\forall x >> 0)\neg(\forall y >> x)((x,y) \notin A).$$
$$(\forall x >> 0)\neg(\forall y >> x)((x,y) \notin B).$$

Again by o-minimality,

$$(\forall x >> 0)(\forall y >> x)((x,y) \in A).$$
$$(\forall x >> 0)(\forall y >> x)((x,y) \in B).$$

This violates $A \cap B = \emptyset$.

For $A \subseteq \mathfrak{R}^2$, let $\mathrm{rev}(A) = \{(x,y): (y,x) \in A\}$.

We now claim that for any three pairwise disjoint semialgebraic $A,B,C \subseteq \mathfrak{R}^2$,

A and rev(A) is small; or
B and rev(B) is small; or
C and rev(C) is small.

To see this, by the previous claim, among every pair of sets drawn from A,B,C, at least one is small. Hence at least two among A,B,C are small. By symmetry, assume A,B are small. Note that rev(A),rev(B) are disjoint and semialgebraic. Hence rev(A) is small or rev(B) is small. This establishes the claim.

For the first claim of the Theorem, let $f:\mathfrak{R}^2 \to \mathfrak{R}$ be semialgebraic. Consider $f^{-1}(0),f^{-1}(1),\ldots$. These sets are semialgebraic and pairwise disjoint. By the above, we see that there exist infinitely many i such that $f^{-1}(i)$ and $\mathrm{rev}(f^{-1}(i))$ are small.

Also note that for all $i \geq 0$, $f^{-1}(i)$ contains all (x,x), x sufficiently large, or excludes all (x,x), x sufficiently large. Because of mutual disjointness, all but at most one $f^{-1}(i)$ has the property that it excludes all (x,x), x sufficiently large.

It is now clear that we can fix i such that

$$f^{-1}(i) \text{ is small.}$$
$$\mathrm{rev}(f^{-1}(i)) \text{ is small.}$$
$$f^{-1}(i) \text{ excludes } (x,x),\ x \text{ sufficiently large.}$$

Let $B = f^{-1}(i)$. We now construct an unbounded open $A \subseteq \mathfrak{R}$ which is disjoint from B.

We have

$$(\forall x >> 0)(\forall y >> x)$$
$$(x,y) \notin B \wedge (y,x) \notin B \wedge (x,x) \notin B.$$

Fix $b > 0$ such that

$$(\forall x \geq b)(\forall y >> x)$$
$$(x,y) \notin B \wedge (y,x) \notin B \wedge (x,x) \notin B.$$

Let $f:(b,\infty) \to \mathfrak{R}$ be semialgebraic such that

$$1)\ (\forall x \geq b)(\forall y \geq f(x))$$
$$(x,y) \notin B \wedge (y,x) \notin B \wedge (x,x) \notin B \wedge f(x) > x.$$

Then f is eventually strictly increasing. Let f be strictly increasing on $[c,\infty)$, $c > b$.

We now define real numbers $c = c_0 < c_1 < \ldots$ as follows.

Define $c_0 = c$. Suppose $c_0 < \ldots < c_i$ have been defined, $i \geq 0$. Define $c_{i+1} = f(c_i)+1$.

For all $i \geq 0$, let $\varepsilon(i) \in (0,1)$ be so small that $B \cap (c_i,c_i+\varepsilon(i))^2 = \emptyset$. We can find $\varepsilon(i)$ since the ordered pair $(c_i,c_i) \notin B$ and B is closed.

By 1), for all $0 \leq i < j$ and $x \in (c_i, c_i+1)$, $y \in (c_j, c_j+1)$, we have $(x,y) \notin B \wedge (y,x) \notin B \wedge (y,y) \notin B$. Hence B is disjoint from A^2, where $A = (c_0, c_0+\varepsilon(0)) \cup (c_1, c1+\varepsilon(1)) \cup \ldots$ is an unbounded open set.

For the second claim of the Theorem, let A be a dense open subset of $\mathfrak{R}$. It suffices to show that $A-A = \mathfrak{R}$. Let $x \in \mathfrak{R}$ and $[a,b] \subseteq A$, $a < b$. Since A is dense, let $y \in [a+x, b+x]$, $y \in A$. Then $y-x \in [a,b]$, and so $y-x \in A$. Hence $x = y-(y-x)$ demonstrates that $x \in A-A$.

For the final claim of the Theorem, let $f:\mathfrak{R}^2 \rightarrow \mathfrak{R}$ be given by $f(x,y) = x \sin(xy)$. Then f is real analytic. Let A be an unbounded open subset of $\mathfrak{R}$ and $z \in \mathfrak{R}$. Let $(a,b) \subseteq A$, $a < b$. Choose $x \in A$ with $|x| > |z|$ so large that $(xa,xb) \cup (xb,xa)$ contains a closed interval of length 2π. Then as y varies in (a,b), the quantity $\sin(xy)$ takes on any value from -1 to 1. Hence as y varies in (a,b), $x \sin(xy)$ takes on any value from $-|x|$ to $|x|$. In particular, it takes on the value z. Therefore $fA = \mathfrak{R}$. QED

Note that in the above development, no distinction between $BFCN^*(\mathfrak{R}^*,\mathfrak{R})$ and $SAFCN(\mathfrak{R}^*,\mathfrak{R})$ has arisen. Also no distinction between $C^1FCN(\mathfrak{R}^*,\mathfrak{R})$, $C^\infty FCN(\mathfrak{R}^*,\mathfrak{R})$, $RAFCN(\mathfrak{R},\mathfrak{R})$, and $CSAFCN(\mathfrak{R}^*,\mathfrak{R})$ has arisen. We suspect that important distinctions will arise as BRT is further developed.

We provide a tabular display of the results in Theorems 1.4.5-1.4.9. We first transcribed the information contained there, where + means that the Thin Set Theorem holds, - means that the Thin Set Theorem fails, and ? means that the Thin Set Theorem is independent of ZFC. We then filled in the remaining entries by immediate inference using obvious inclusion relations between the various classes of functions and the various classes of sets.

	cSUB	UNCLSUB	NOPSUB	UNOPSUB	DEOPSUB	FMOPESUB	CCOPSUB	FCSUB	≤1CSUB
$FCN(\mathfrak{R},\mathfrak{R})$	+	–	–	–	–	–	–	–	–
$BFCN(\mathfrak{R},\mathfrak{R})$	+	+	–	–	–	–	–	–	–
$CFCN(\mathfrak{R},\mathfrak{R})$	+	+	+	+	+	+	–	–	–
$C^1FCN(\mathfrak{R},\mathfrak{R})$	+	+	+	+	+	+	+	–	–
$C^\infty FCN(\mathfrak{R},\mathfrak{R})$	+	+	+	+	+	+	+	–	–
$RAFCN(\mathfrak{R},\mathfrak{R})$	+	+	+	+	+	+	+	–	–
$SAFCN(\mathfrak{R},\mathfrak{R})$	+	+	+	+	+	+	+	+	–
$CSAFCN(\mathfrak{R},\mathfrak{R})$	+	+	+	+	+	+	+	+	+
$FCN(\mathfrak{R}^*,\mathfrak{R})$	?	–	–	–	–	–	–	–	–

$FCN(\Re^2,\Re)$	?	−	−	−	−	−	−	−	−
$BFCN(\Re^*,\Re)$	+	+	−	−	−	−	−	−	−
$CFCN(\Re^*,\Re)$	+	+	+	−	−	−	−	−	−
$C^1FCN(\Re^*,\Re)$	+	+	+	−	−	−	−	−	−
$C^\infty FCN(\Re^*,\Re)$	+	+	+	−	−	−	−	−	−
$RAFCN(\Re^*,\Re)$	+	+	+	−	−	−	−	−	−
$RAFCN(\Re^2,\Re)$	+	+	+	−	−	−	−	−	−
$SAFCN(\Re^*,\Re)$	+	+	−	−	−	−	−	−	−
$SAFCN(\Re^3,\Re)$	+	+	−	−	−	−	−	−	−
$SAFCN(\Re^2,\Re)$	+	+	+	+	−	−	−	−	−
$CSAFCN(\Re^*,\Re)$	+	+	+	−	−	−	−	−	−
$CSAFCN(\Re^3,\Re)$	+	+	+	−	−	−	−	−	−
$CSAFCN(\Re^2,\Re)$	+	+	+	+	−	−	−	−	−