PREFACE

The standard axiomatization of mathematics is given by the formal system ZFC, which is read "Zermelo Frankel set theory with the axiom of choice".

The vast majority of mathematical proofs fit easily into the ZFC formalism. ZFC has stood the test of time.

However, a long list of mathematically natural statements of an abstract set theoretic nature have been shown to be undecided (neither provable nor refutable) in ZFC, starting with the pioneering work of Kurt Gödel and Paul J. Cohen concerning Cantor's continuum hypothesis.

Yet these statements involve general notions that are uncharacteristic of normal mathematical statements. The unprovability and unrefutability from ZFC depends on this uncharacteristic generality. For example, if we remove this uncharacteristic generality from Cantor's continuum hypothesis, we obtain a well known theorem of Aleksandrov and Hausdorff (see [Al16] and [Hau16]).

Already as a student at MIT in the mid 1960s, I recognized the critical issue of whether ZFC suffices to prove or refute all concrete mathematically natural statements. Here concreteness refers to the lack of involvement of objects of a distinctly pathological nature. In particular, the finite, the discrete, and the continuous (on nice spaces) are generally considered concrete - although, generally speaking, only the finite is beyond reproach.

From my discussions then with faculty and fellow students, it became clear that according to conventional wisdom, the Incompleteness Phenomena was confined to questions of an inherently set theoretic nature. The incompleteness would not appear if this uncharacteristic generality is removed.

According to conventional wisdom, reasonably well motivated problems in relatively concrete standard mathematical settings can be settled with the usual axioms for mathematics (as formalized by ZFC). The difficulties associated with such problems are inherently mathematical and not "logical" or "foundational".

It was already clear to me at that time that despite the great depth and beauty of the ongoing breakthroughs in set theory regarding the continuum hypothesis and many other tantalizing set theoretic problems, the long range impact and significance of ongoing investigations in the foundations of mathematics is going to depend greatly on the extent to which the Incompleteness Phenomena touches normal concrete mathematics. This perception was confirmed in my first few years out of school at Stanford University with further discussions with mathematics faculty, including Paul J. Cohen.

Yet I was confronted with a major strategic decision early in my career concerning how, or even whether, to investigate this issue of Concrete Mathematical Incompleteness.

The famous incompleteness results of Gödel and Cohen involving the Axiom of Choice (over ZF) and the Continuum Hypothesis (over ZFC), involved problems that had previously been formulated. In fact, the Axiom of Choice and the Continuum Hypothesis were widely offered up as candidates for Incompleteness.

Yet there were no candidates for Concrete Mathematical Incompleteness from ZFC being offered. In fact, to this day, no candidates for Concrete Mathematical Incompleteness have arisen from the natural course of mathematics.

In fact, it still seems rather likely that all concrete problems that have arisen thus far from the natural course of mathematics can be proved or refuted within ZFC.

So what can be the rationale for pursuing a search for Concrete Mathematical Incompleteness?

We offer two rationales for pursuing Concrete Mathematical Incompleteness. One is the presence of Concrete Mathematical Incompleteness in the weaker sense of being independent of significant fragments of ZFC. Since the vast bulk of mathematical activity involves insignificant fragments of ZFC, examples where significant fragments of ZFC are required is significant from the point of view of the foundations of mathematics.

In fact, we do have a rather convincing example of Concrete Mathematical Incompleteness arising from an existing - in fact celebrated - mathematical theorem. This is the theorem of J.B. Kruskal about finite trees. See the detailed discussion in section 0.9B of the Introduction. The story continues with the also celebrated Graph Minor Theorem, as discussed in section 0.10B of the Introduction.

Once the ice is broken with the Concrete Mathematical Incompleteness of existing celebrated theorems, it appears inevitable to consider examples of Concrete Mathematical Incompleteness from significant fragments of ZFC that are in various senses "almost existing mathematical theorems" or "close to existing mathematical theorems" or "simple modifications of existing mathematical theorems". Most of the Introduction is devoted to a detailed discussion of such examples.

The second rationale for pursuing Concrete Mathematical Incompleteness preserves ZFC as the ambitious target. The idea is that normal mathematical activity up to now represents only an infinitesimal portion of eventual mathematical activity. Even if current mathematical activity does not give rise to Concrete Mathematical Incompleteness from ZFC, this is a very poor indication of whether this will continue to be the case, particularly far out into the future.

These considerations give rise to the prospect of uncovering mathematical areas of the future, destined to arise along many avenues, that are replete with Concrete Mathematical Incompleteness from ZFC.

We believe that Boolean Relation Theory is such a field from the future. Most of this book is devoted to Concrete Mathematical Incompleteness from ZFC that arises in Boolean Relation Theory.

We anticipate that further development of BRT will uncover additional connections with concrete mathematical activity - strengthening the argument that it is a field from the future - as well as additional Concrete Mathematical Incompleteness from ZFC.

While completing this book, we have continued the search for additional Concrete Mathematical Incompleteness that opens up new connections with normal mathematics. These new developments - which have yet to be prepared for publication - are discussed in sections 0.14D - 0.14I. They suggest a general structure theory for maximal objects which can, and can only be carried out with the use of large cardinal hypotheses (or their consistency with ZFC). The extent to which these new developments invade mathematics remains to be seen.