2.2. EBRT, IBRT in A,fA.

This section is intended to be a particularly gentle introduction to BRT classification theory. It is wholly subsumed by section 2.3.

Recall the five main BRT settings introduced at the beginning of this Chapter: (SD,INF), (ELG,INF), (MF,INF), (ELG ∩ SD,INF), (EVSD,INF).

We begin with the BRT fragments α =

EBRT in A,fA on these five BRT settings.

As discussed in sections 1.1 and 2.1, classification of these BRT fragments amounts to a determination of the true α assertions, which take the form

$$1)\ (\forall f \in V)(\exists A \in K)(\varphi)$$

where φ is an α equation (since we are in the environment EBRT).

As discussed in sections 1.1 and 2.1, we work, equivalently, with the α statements, which take the form

$$1')\ (\forall f \in V)(\exists A \in K)(S)$$

where S is an α format, interpreted conjunctively.

Recall that in EBRT, S is correct if and only if 1') holds. S is incorrect if and only if 1') fails.

In this case of EBRT in A,fA, the number of elementary inclusions is 4, and the number of formats is 16.

Since 16 is so small, we might as well list all of the formats S. It is most convenient to list the formats S in increasing order of their cardinality - which is 0-4.

The four A,fA elementary inclusions are as follows. See Definition 1.1.36. (These do not depend on the BRT environment or BRT setting).

$$A \cap fA = \emptyset.$$
$$A \cup fA = U.$$
$$A \subseteq fA.$$

fA ⊆ A.

According to Definition 1.1.13 of the universal set U in BRT settings, we see that on our five BRT settings, U is N.

Before beginning this tabular EBRT classification, we organize the nontrivial mathematical facts that we will use.

THEOREM 2.2.1. Let f ∈ EVSD and E ⊆ A ⊆ N, where E is finite, A is infinite, and E ∩ fE = ∅. Also let D ⊆ N be infinite. There exists infinite B such that E ⊆ B ⊆ A, B ∩ fB = ∅, and neither A nor D are subsets of B ∪ fB. Moreover, this is provable in RCA_0.

Proof: Let f,E,A,D be as given. Let n ∈ D be such that n > max(E ∪ fE), and |x| ≥ n → f(x) > |x|. Let t > n, t ∈ A. We define an infinite strictly increasing sequence $n_1 < n_2$ ... by induction as follows.

Define n_1 = min{m ∈ A: m > t}. Suppose $n_1 < ... < n_k$ have been defined, k ≥ 1. Define n_{k+1} to be the least element of A that is greater than n_k and all elements of f(E ∪ {n_1,...,n_k}).

Let B = E ∪ {n_1,n_2,...} ⊆ A. Clearly B ∩ fB = ∅. Also n,t ∉ B, and so A,D are not subsets of B ∪ fB. QED

In the applications of Theorem 2.2.1 to the tabular EBRT classification below, we can ignore E,A,D. We just use that for all f ∈ EVSD, there exists infinite B ⊆ N such that B ∩ fB = ∅.

Here is the other fact that we need.

COMPLEMENTATION THEOREM. For all f ∈ SD there exists A ∈ INF such that fA = N\A.

We proved the Complementation Theorem in section 1.3 within RCA_0.

A,fA FORMAT OF CARDINALIY 0
EBRT

The empty format is obviously correct, on all five BRT settings.

A,fA FORMATS OF CARDINALITY 1
EBRT

1.1. A ∩ fA = ∅.
1.2. A ∪ fA = U. Correct on all five. Set A = N.
1.3. A ⊆ fA. Incorrect on all five. Set f(x) = 2x+1.
1.4. fA ⊆ A. Correct on all five. Set A = N.

A,fA FORMATS OF CARDINALITY 2
EBRT

2.1. A ∩ fA = ∅, A ∪ fA = U. Equivalent to fA = U\A on all five.
2.2. A ∩ fA = ∅, A ⊆ fA. Incorrect on all five. Contains 1.3.
2.3. A ∩ fA = ∅, fA ⊆ A. Incorrect on all five.
2.4. A ∪ fA = U, A ⊆ fA. Incorrect on all five. Contains 1.3.
2.5. A ∪ fA = U, fA ⊆ A. Correct on all five. Set A = U.
2.6. A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.3.

A,fA FORMATS OF CARDINALITY 3
EBRT

3.1. A ∩ fA = ∅, A ∪ fA = U, A ⊆ fA. Incorrect on all five. Contains 1.3.
3.2. A ∩ fA = ∅, A ∪ fA = U, fA ⊆ A. Incorrect on all five. Contains 2.3.
3.3. A ∩ fA = ∅, A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.3.
3.4. A ∪ fA = U, A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.3.

A,fA FORMAT OF CARDINALITY 4
EBRT

4.1. A ∩ fA = ∅, A ∪ fA = U, A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.3.

We now list all of the formats whose status has not been determined. We use any stated equivalences that hold on all five.

1.1. A ∩ fA = ∅.
2.1. fA = U\A.

We now indicate the status of 1.1, 1.2, for EBRT in A,fA on each of our five main BRT settings.

We heavily use the fact that every function in our five main BRT settings, with the exception of MF, has infinite range.

EBRT in A,fA on (SD,INF)/(ELG ∩ SD,INF)

1.1. A ∩ fA = ∅. Correct on both. See Theorem 2.2.1.
2.1. fA = U\A. Correct on both. The Complementation Theorem.

EBRT in A,fA on (ELG,INF)/(EVSD,INF)

1.1. A ∩ fA = ∅. Correct on both. See Theorem 2.2.1.
2.1. fA = U\A. Incorrect on both. Let f(x) = 0 if x = 0; 2x+1 otherwise.

EBRT in A,fA on (MF,INF)

1.1. A ∩ fA = ∅. Incorrect. Let f(x) = x.
2.1. fA = U\A. Incorrect. Let f(x) = x.

We now make a table from our findings. + indicates that the format along left column is α correct, where α is EBRT in A,fA on the setting across the top row. – indicates otherwise.

EBRT in A,fA on:	(SD,INF)	(ELG ∩ SD,INF)	(ELG,INF)	(EVSD,INF)	(MF,INF)
∅	+	+	+	+	+
A ∩ fA = ∅	+	+	+	+	-
A ∪ fA = U	+	+	+	+	+
A ⊆ fA	-	-	-	-	-
fA ⊆ A	+	+	+	+	+
A ∩ fA = ∅, A ∪ fA = U	+	+	-	-	-
A ∩ fA = ∅, A ⊆ fA	-	-	-	-	-
A ∩ fA = ∅, fA ⊆ A	-	-	-	-	-
A ∪ fA = U, A ⊆ fA	-	-	-	-	-
A ∪ fA = U, fA ⊆ A	+	+	+	+	+
A ⊆ fA, fA ⊆ A	-	-	-	-	-
A ∩ fA = ∅, A ∪ fA = U, A ⊆ fA	-	-	-	-	-
A ∩ fA = ∅, A ∪ fA = U, fA ⊆ A	-	-	-	-	-
A ∩ fA = ∅, A ⊆ fA, fA ⊆ A	-	-	-	-	-
A ∪ fA = U, A ⊆ fA, fA ⊆ A	-	-	-	-	-
A ∩ fA = ∅,A ∪ fA = U,A ⊆ fA,fA ⊆ A	-	-	-	-	-

THEOREM 2.2.2. EBRT in A,fA on (SD,INF),(ELG ∩ SD,INF) have the same correct formats (or, equivalently, true statements, or true assertions). So do EBRT in A,fA on (ELG,INF),(EVSD,INF). This is not true of EBRT in A,fA on

any distinct pair of settings among (SD,INF),(ELG,INF),(MF,INF). EBRT in A,fA on all five settings, is RCA_0 secure.

Proof: Immediate from the above tabular classifications and their documentation. This uses the observation that Theorem 2.2.1 and the Complementation Theorem are provable in RCA_0. The counterexamples are very explicit. QED

We now come to IBRT in A,fA on the same five BRT settings. We investigate the assertions

$$2)\ (\forall f \in V)(\exists A \in K)(\varphi)$$

where φ is an α inequation (since we are in the environment IBRT).

As discussed in sections 1.1 and 2.1, we work, equivalently, with the α statements, which take the form

$$2')\ (\exists f \in V)(\forall A \in K)(S)$$

where S is an α format, interpreted conjunctively.

Recall that in IBRT, S is correct if and only if 2') holds. S is incorrect if and only if 2') fails.

We again start with the same four A,fA elementary inclusions, as these do not depend on the environment.

Before beginning this tabular EBRT classification, we organize the nontrivial facts that we will use. Recall the Thin Set Theorem from section 1.4.

THIN SET THEOREM. For all $f \in MF$ there exists $A \in INF$ such that $fA \neq N$.

We also need the following variant.

THIN SET THEOREM (variant). For all $f \in MF$ there exists $A \in INF$ such that $A \cup fA \neq N$.

Proof: We derive this variant from the Thin Set Theorem (over RCA_0). Let $f:N^k \rightarrow N$. Define $g:N^{k+1} \rightarrow N$ by $g(x_1,\ldots,x_{k+1}) = f(x_1,\ldots,x_k)$ if $x_k \neq x_{k+1}$; x_k otherwise. By the Thin Set Theorem, let $A \in INF$, $gA \neq N$. Then $gA = A \cup fA \neq N$. QED

By the above proof, it is clear that the Thin Set Theorem and the Thin Set Theorem (variant) are provably equivalent in RCA_0.

The system ACA' (see Definition 1.4.1) is sufficient to prove the Thin Set Theorem. Here are the four A,fA elementary inclusions.

$$A \cap fA = \emptyset.$$
$$A \cup fA = U.$$
$$A \subseteq fA.$$
$$fA \subseteq A.$$

A,fA FORMAT OF CARDINALIY 0
IBRT

The empty format is obviously correct, on all five BRT settings.

A,fA FORMATS OF CARDINALITY 1
IBRT

1.1. $A \cap fA = \emptyset$. Incorrect on all five. Set A = N.
1.2. $A \cup fA = U$. Incorrect on all five. Thin Set Theorem (variant).
1.3. $A \subseteq fA$.
1.4. $fA \subseteq A$.

A,fA FORMATS OF CARDINALITY 2
IBRT

2.1. $A \cap fA = \emptyset$, $A \cup fA = U$. Incorrect on all five. Contains 1.1.
2.2. $A \cap fA = \emptyset$, $A \subseteq fA$. Incorrect on all five. Contains 1.1.
2.3. $A \cap fA = \emptyset$, $fA \subseteq A$. Incorrect on all five. Contains 1.1
2.4. $A \cup fA = U$, $A \subseteq fA$. Incorrect on all five. Contains 1.2.
2.5. $A \cup fA = U$, $fA \subseteq A$. Incorrect on all five. Contains 1.2.
2.6. $A \subseteq fA$, $fA \subseteq A$. Equivalent to $fA = A$ on all five.

A,fA FORMATS OF CARDINALITY 3
IBRT

3.1. A ∩ fA = ∅, A ∪ fA = U, A ⊆ fA. Incorrect on all five. Contains 1.1.
3.2. A ∩ fA = ∅, A ∪ fA = U, fA ⊆ A. Incorrect on all five. Contains 1.1.
3.3. A ∩ fA = ∅, A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
3.4. A ∪ fA = U, A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.2.

A,fA FORMAT OF CARDINALITY 4
IBRT

A ∩ fA = ∅, A ∪ fA = U, A ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.

These are the only formats whose status has not been determined. We use equivalences that hold on all five.

1.3. A ⊆ fA.
1.4. fA ⊆ A.
2.6. fA = A.

We now indicate the status of 1.3, 1.4, 1.6, for IBRT in A,fA on each of our five main BRT settings.

IBRT in A,fA on (SD,INF),(ELG ∩ SD,INF),(ELG,INF),(EVSD,INF)

1.3. A ⊆ fA. Incorrect on all four. Theorem 2.2.1.
1.4. fA ⊆ A. Incorrect on all four. Theorem 2.2.1.
2.6. fA = A. Incorrect on all four. Theorem 2.2.1.

IBRT in A,fA on (MF,INF)

1.3. A ⊆ fA. Correct. Set f(x) = x.
1.4. fA ⊆ A. Correct. Set f(x) = x.
2.6. fA = A. Correct. Set f(x) = x.

Recall that the instances of 2’) are in dual form. I.e., they are the negations of the IBRT in A,fA assertions. In particular, the Thin Set Theorem and the Thin Set Theorem (variant) are assertions in IBRT in A,fA on (MF,INF), and therefore negations of statements in IBRT in A,fA on (MF,INF).

We now make a table from our findings. + indicates that the format along left column is α correct, where α is IBRT in

A,fA on the setting across the top row. – indicates otherwise.

IBRT in A,fA on:	(SD,INF)	(ELG ∩ SD,INF)	(ELG,INF)	(EVSD,INF)	(MF,INF)
∅	+	+	+	+	+
A ∩ fA = ∅	-	-	-	-	-
A ∪ fA = U	-	-	-	-	-
A ⊆ fA	-	-	-	-	+
fA ⊆ A	-	-	-	-	+
A ∩ fA = ∅, A ∪ fA = U	-	-	-	-	-
A ∩ fA = ∅, A ⊆ fA	-	-	-	-	-
A ∩ fA = ∅, fA ⊆ A	-	-	-	-	-
A ∪ fA = U, A ⊆ fA	-	-	-	-	-
A ∪ fA = U, fA ⊆ A	-	-	-	-	-
A ⊆ fA, fA ⊆ A	-	-	-	-	+
A ∩ fA = ∅, A ∪ fA = U, A ⊆ fA	-	-	-	-	-
A ∩ fA = ∅, A ∪ fA = U, fA ⊆ A	-	-	-	-	-
A ∩ fA = ∅, A ⊆ fA, fA ⊆ A	-	-	-	-	-
A ∪ fA = U, A ⊆ fA, fA ⊆ A	-	-	-	-	-
A ∩ fA = ∅,A ∪ fA = U,A ⊆ fA,fA ⊆ A	-	-	-	-	-

THEOREM 2.2.3. For IBRT in A,fA on (SD,INF),(ELG ∩ SD,INF), (ELG,INF),(EVSD,INF), the only correct format is ∅. This is not true of IBRT in A,fA on (MF,INF). IBRT in A,fA on each of (SD,INF),(ELG ∩ SD,INF),(ELG,INF),(EVSD,INF) is RCA_0 secure. IBRT in A,fA on (MF,INF) is ACA' secure, but not ACA_0 secure. Every correct format in A,fA on (MF,INF) is RCA_0 correct. We can replace ACA' here by RCA_0 + Thin Set Theorem.

Proof: The first two claims are immediate from the given tabular classifications. For the third claim, it suffices to verify that the incorrectness of 1.1,1.2,1.3,1.4,2.6 on these four settings is provable in RCA_0. For 1.1, this is trivial. For 1.3, 1.4, 2.6, this is from the provability of Theorem 2.2.1 in RCA_0. For 1.2, this is also from the provability of Theorem 2.2.1 in RCA_0.

For the fourth claim, note that all correctness determinations in IBRT in A,fA on (MF,INF) were given in RCA_0, and all incorrectness determinations in α = IBRT in A,fA on (MF,INF) were given in RCA_0 + Thin Set Theorem (variant). Thus α is RCA_0 + Thin Set Theorem (variant) secure, and hence ACA' secure. Since the incorrectness of 1.2 in α is equivalent, over RCA_0, to Thin Set Theorem (variant), α is not ACA_0 secure. This is because of the unprovability of the Thin Set Theorem in ACA_0 (see [FS00], [CGHJ05]).

For the fifth claim, α is RCA_0 + Thin Set Theorem secure, because of the proof given above of Thin Set Theorem (variant) from Thin Set Theorem (over RCA_0). QED

An interesting issue is the effect of the arity of the functions. The classes SD, ELG ∩ SD, ELG, EVSD, and MF use functions of every arity k ≥ 1.

LEMMA 2.2.4. The Thin Set Theorem (variant) for exponent 1 is provable in RCA_0.

Proof: Let f:N → N. If {n: f(n) = 0} is infinite then set A = {n: f(n) = 0}. If not, let n > 0 be such that f is nonzero on [n,∞). Set A = [n,∞). Then A ∪ fA ≠ N. QED

DEFINITION 2.2.1. For k ≥ 1, let SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], MF[k] be the restrictions of SD, ELG ∩ SD, ELG, EVSD, MF to functions whose domain is N^k.

THEOREM 2.2.5. Let k ≥ 1. EBRT in A,fA on SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], MF[k], and IBRT in A,fA on SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], are RCA_0 secure. IBRT in A,fA on MF[k] is ACA_0 secure. IBRT in A,fA on MF[1] is RCA_0 secure. EBRT and IBRT in A,fA on SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], MF[k] have the same correct formats as EBRT and IBRT in SD, ELG ∩ SD, ELG, EVSD, MF, respectively.

Proof: An examination of the arguments immediately reveals that all of the incorrectness determinations given for EBRT involve unary functions only, and all of the correctness determinations given for IBRT also involve unary functions only. We can obviously pad these unary functions as k-ary functions. IBRT in A,fA on MF[k] is ACA_0 secure since the Thin Set Theorem (variant) is provable in ACA_0 for k-ary functions, using the infinite Ramsey theorem for k-tuples. By Lemma 2.2.4, IBRT in A,fA on MF[1] is RCA_0 secure. QED