2.3. EBRT, IBRT in A,fA,fU.

We redo section 2.2 for the signature A,fA,fU, with the same five BRT settings (SD,INF), (ELG ∩ SD,INF), (ELG,INF), (EVSD,INF), (MF,INF).

After we treat these five BRT settings, we then treat the five corresponding unary BRT settings (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF). These are the same except that we restrict to the 1-ary functions only. There is quite a lot of difference between the unary settings and the multivariate settings; this was not the case in section 2.2, with just A,fA.

We begin with EBRT in A,fA,fU. The 8 A,fA,fU pre elementary inclusions are as follows (see Definition 1.1.35).

A ∩ fA ∩ fU = ∅.
A ∪ fA ∪ fU = U.
A ⊆ fA ∪ fU.
fA ⊆ A ∪ fU.
fU ⊆ A ∪ fA.
A ∩ fA ⊆ fU.
A ∩ fU ⊆ fA.
fA ∩ fU ⊆ A.

The 6 A,fA,fU elementary inclusions are as follows (see Definition 1.1.36).

A ∩ fA = ∅.
A ∪ fU = U.
A ⊆ fU.
fU ⊆ A ∪ fA.
A ∩ fU ⊆ fA.
fA ⊆ A.

We will use Theorem 2.2.1, and the Complementation Theorem from section 1.3. In fact, we need the following strengthening of the Complementation Theorem.

THEOREM 2.3.1. Let f ∈ SD and B ∈ INF. There exists A ∈ INF, A ⊆ B, such that A ∩ fA = ∅ and B ⊆ A ∪ fA. Moreover, this is provable in RCA_0.

Proof: Let f,B be as given. We inductively define A ⊆ B as follows. Suppose the elements of A from 0,1,...,n-1 have been defined, n ≥ 0. We put n in A if and only if n ∈ B and n is not the value of f at arguments from A less than n. Then A is as required, using f ∈ SD. QED

THEOREM 2.3.2. For all f ∈ EVSD there exists A ∈ INF such that A ∩ fA = ∅, A ∪ fN = N. Moreover, this is provable in RCA_0.

Proof: Let f ∈ EVSD be k-ary. Let n be such that |x| ≥ n → f(x) ≥ |x|. We define A inductively. First put [0,n]\fN in A. For m > n, put m in A if and only if m = f(x) for no |x| < m. Then [n+1,∞) ⊆ A ∪ fA. Also [0,n] ⊆ A ∪ fA. Hence A ∪ fN = N. Suppose m ∈ A ∩ fA. If m ≤ n then by construction, m ∈ [0,n]\fN, contradicting m ∈ fA. Hence m > n. Let m = f(x), x ∈ A^k. If |x| ≥ m then f(x) > |x| ≥ m, which is a contradiction. Hence |x| < m, and so m ∉ A by construction. This contradicts m ∈ A. QED

THEOREM 2.3.3. Let k ≥ 2. There exists k-ary f ∈ ELG ∩ SD such that N\fN = {0}. There exists k-ary f ∈ ELG such that fN = N.

Proof: For all n ≥ 1, let $f_n:[2^n,2^{n+1})^k \to [2^{n+1},2^{n+2})$ be onto. Let f be the union of the f_n extended as follows. For x not yet defined, set f(x) = 1 if |x| = 0; 2 if |x| = 1; 3 if |x| = 2; 2|x| if |x| ≥ 3. Then fN = N\{0} and f ∈ ELG ∩ SD. Let g be the union of the f_n extended as follows. For x not yet defined, set f(x) = 0 if |x| = 0; 1 if |x| = 1; 2 if |x| = 2; 3 if |x| = 3; 2|x| if |x| ≥ 4. Then fN = N and f ∈ ELG. QED

SETTINGS: (SD,INF), (ELG ∩ SD,INF),
(ELG,INF), (EVSD,INF), (MF,INF).

A,fA,fU FORMAT OF CARDINALITY 0
EBRT

The empty format is obviously correct, for all five BRT settings.

A,fA,fU FORMATS OF CARDINALITY 1
EBRT

1.1. A ∩ fA = ∅.
1.2. A ∪ fU = U. Correct on all five. Set A = N.
1.3. A ⊆ fU.
1.4. fU ⊆ A ∪ fA. Correct on all five. Set A = N.
1.5. A ∩ fU ⊆ fA. Correct on all five. Set A = N.

1.6. $fA \subseteq A$. Correct on all five. Set $A = N$.

A,fA,fU FORMATS OF CARDINALITY 2
EBRT

2.1. $A \cap fA = \emptyset$, $A \cup fU = U$.
2.2. $A \cap fA = \emptyset$, $A \subseteq fU$.
2.3. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$.
2.4. $A \cap fA = \emptyset$, $A \cap fU \subseteq fA$. Equivalent on all five to $A \cap fU = \emptyset$. Incorrect on all five. Theorem 2.3.3.
2.5. $A \cap fA = \emptyset$, $fA \subseteq A$. Equivalent on all five to $fA = \emptyset$. Incorrect on all five using any f.
2.6. $A \cup fU = U$, $A \subseteq fU$. Equivalent on all five to $fU = U$. Incorrect on all five. Set $rng(f) \neq N$.
2.7. $A \cup fU = U$, $fU \subseteq A \cup fA$. Correct on all five. Set $A = N$.
2.8. $A \cup fU = U$, $A \cap fU \subseteq fA$. Correct on all five. Set $A = N$.
2.9. $A \cup fU = U$, $fA \subseteq A$. Correct on all five. Set $A = N$.
2.10. $A \subseteq fU$, $fU \subseteq A \cup fA$.
2.11. $A \subseteq fU$, $A \cap fU \subseteq fA$. Equivalent on all five to $A \subseteq fA$. Incorrect on all five. Set $f(x) = 2x+1$.
2.12. $A \subseteq fU$, $fA \subseteq A$.
2.13. $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Correct on all five. Set $A = N$.
2.14. $fU \subseteq A \cup fA$, $fA \subseteq A$. Correct on all five. Set $A = N$.
2.15. $A \cap fU \subseteq fA$, $fA \subseteq A$. Correct on all five. Set $A = N$.

A,fA,fU FORMATS OF CARDINALITY 3
EBRT

3.1. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \subseteq fU$. Incorrect on all five. Contains 2.6.
3.2. $A \cap fA = \emptyset$, $A \cup fU = U$, $fU \subseteq A \cup fA$. Equivalent to $A \cap fA = \emptyset$, $A \cup fA = U$ on all five.
3.3. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 2.4.
3.4. $A \cap fA = \emptyset$, $A \cup fU = U$, $fA \subseteq A$. Incorrect on all five. Contains 2.5.
3.5. $A \cap fA = \emptyset$, $A \subseteq fU$, $fU \subseteq A \cup fA$.
3.6. $A \cap fA = \emptyset$, $A \subseteq fU$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 2.11.
3.7. $A \cap fA = \emptyset$, $A \subseteq fU$, $fA \subseteq A$. Incorrect on all five. Contains 2.5.
3.8. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 2.4.

3.9. A ∩ fA = ∅, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
3.10. A ∩ fA = ∅, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
3.11. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on all five. Contains 2.6.
3.12. A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
3.13. A ∪ fU = U, A ⊆ fU, fA ⊆ A. Incorrect on all five. Contains 2.6.
3.14. A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Correct on all five. Set A = N.
3.15. A ∪ fU = U, fU ⊆ A ∪ fA, fA ⊆ A. Correct on all five. Set A = N.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Correct on all five. Set A = N.
3.17. A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.11.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A.
3.19. A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.11.
3.20. fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Correct on all five. Set A = N.

A,fA,fU FORMATS OF CARDINALITY 4
EBRT

4.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on all five. Contains 2.6.
4.2. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
4.3. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.4. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.4.
4.5. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.6. A ∩ fA = ∅, A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.7. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.11.
4.8. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.9. A ∩ fA = ∅, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.10. A ∩ fA = ∅, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.

4.11. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
4.12. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.6.
4.13. A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.6.
4.14. A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Correct on all five. Set A = N.
4.15. A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.11.

A,fA,fU FORMATS OF CARDINALITY 5
EBRT

5.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
5.2. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.3. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.4. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.6. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.6.

A,fA,fU FORMATS OF CARDINALITY 6
EBRT

6.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.

We now list all of the formats whose status has not been determined. We use any stated equivalences that hold on all five.

1.1. A ∩ fA = ∅.
1.3. A ⊆ fU.
2.1. A ∩ fA = ∅, A ∪ fU = U.
2.2. A ∩ fA = ∅, A ⊆ fU.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA.
2.10. A ⊆ fU, fU ⊆ A ∪ fA.
2.12. A ⊆ fU, fA ⊆ A.
3.2. A ∩ fA = ∅, A ∪ fA = U.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A.

We now settle the status of each of these formats on the various settings.

EBRT in A,fA,fU on (SD,INF),(ELG ∩ SD,INF)

1.1. A ∩ fA = ∅. Correct on both. See Theorem 2.2.1.
1.3. A ⊆ fU. Correct on both. Set A = fN.
2.1. A ∩ fA = ∅, A ∪ fU = U. Correct on both. The Complementation Theorem (section 1.3).
2.2. A ∩ fA = ∅, A ⊆ fU. Correct on both. Theorem 2.2.1.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Correct on both. The Complementation Theorem.
2.10. A ⊆ fU, fU ⊆ A ∪ fA. Correct on both. Set A = fN.
2.12. A ⊆ fU, fA ⊆ A. Correct on both. Set A = fN.
3.2. A ∩ fA = ∅, A ∪ fA = U. Correct on both. Complementation Theorem.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Correct on both. Theorem 2.3.1 with B = fU.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Correct on both. Set A = fN.

EBRT in A,fA,fU on (ELG,INF),(EVSD,INF)

1.1. A ∩ fA = ∅. Correct on both. See Theorem 2.2.1.
1.3. A ⊆ fU. Correct on both. Set A = fN.
2.1. A ∩ fA = ∅, A ∪ fU = U. Correct on both. Theorem 2.3.2.
2.2. A ∩ fA = ∅, A ⊆ fU. Correct on both. Theorem 2.2.1.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Incorrect on both. Set f(x) = 2x.
2.10. A ⊆ fU, fU ⊆ A ∪ fA. Correct on both. Set A = fN.
2.12. A ⊆ fU, fA ⊆ A. Correct on both. Set A = fN.
3.2. A ∩ fA = ∅, A ∪ fA = U. Incorrect on both. Set f(x) = 2x.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on both. Set f(x) = 2x.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Correct on both. Set A = fN.

From the above, we see a difference between (SD,INF) and (EVSD,INF) with regard to 2.3, 3.2, 3.5.

EBRT in A,fA,fU on (MF,INF)

1.1. A ∩ fA = ∅. Incorrect. Set f(x) = x.
1.3. A ⊆ fU. Incorrect. Set f(x) = 0.

2.1. A ∩ fA = ∅, A ∪ fU = U. Incorrect. Set f(x) = x.
2.2. A ∩ fA = ∅, A ⊆ fU. Incorrect. Set f(x) = x.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Incorrect. Set f(x) = x.
2.10. A ⊆ fU, fU ⊆ A ∪ fA. Incorrect. Set f(x) = 0.
2.12. A ⊆ fU, fA ⊆ A. Correct on both. Set A = fN. Incorrect. Set f(x) = 0.
3.2. A ∩ fA = ∅, A ∪ fA = U. Incorrect. Set f(x) = x.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect. Set f(x) = x.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect. Set f(x) = 0.

THEOREM 2.3.4. EBRT in A,fA,fU on (SD,INF),(ELG ∩ SD,INF) have the same correct formats. So do EBRT in A,fA,fU on (ELG,INF),(EVSD,INF). This is not true of EBRT in A,fA,fU on any distinct pair of settings among (SD,INF),(ELG,INF),(MF,INF). EBRT in A,fA on all five settings, is RCA_0 secure.

Proof: Immediate from the above tabular classifications and their documentation. Format 2.3 provides a difference between A,fA,fU on (SD[1],INF) and (ELG[1],INF), and on (SD,INF) and (MF,INF). Format 1.3 proves a difference between A,fA,fU on (ELG,INF) and (MF,INF). QED

We now turn to IBRT in A,fA,fU on the same five BRT settings.

We will use the Thin Set Theorem (variant) from section 2.2, as well as Theorem 2.2.1, and previous results of this section.

SETTINGS: (SD,INF), (ELG ∩ SD,INF), (ELG,INF), (EVSD,INF), (MF,INF).

A,fA,fU FORMAT OF CARDINALITY 0
IBRT

The empty format is obviously correct, for all five BRT settings.

A,fA,fU FORMATS OF CARDINALITY 1
IBRT

1.1. A ∩ fA = ∅. Incorrect on all five. Set A = N.
1.2. A ∪ fU = U.
1.3. A ⊆ fU.
1.4. fU ⊆ A ∪ fA.

1.5. A ∩ fU ⊆ fA.
1.6. fA ⊆ A.

A,fA,fU FORMATS OF CARDINALITY 2
IBRT

2.1. A ∩ fA = ∅, A ∪ fU = U. Incorrect on all five. Contains 1.1.
2.2. A ∩ fA = ∅, A ⊆ fU. Incorrect on all five. Contains 1.1.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Incorrect on all five. Contains 1.1.
2.4. A ∩ fA = ∅, A ∩ fU ⊆ fA. Incorrect on all five. Contains 1.1.
2.5. A ∩ fA = ∅, fA ⊆ A. Incorrect on all five. Contains 1.1.
2.6. A ∪ fU = U, A ⊆ fU. Equivalent on all five to fU = U.
2.7. A ∪ fU = U, fU ⊆ A ∪ fA. Equivalent on all five to A ∪ fA = U. Incorrect on all five. Thin Set Theorem (variant).
2.8. A ∪ fU = U, A ∩ fU ⊆ fA.
2.9. A ∪ fU = U, fA ⊆ A.
2.10. A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on all five. Suppose fN ≠ N. Set A = N. Suppose fN = N. Thin Set Theorem (variant).
2.11. A ⊆ fU, A ∩ fU ⊆ fA. Equivalent on all five to A ⊆ fA.
2.12. A ⊆ fU, fA ⊆ A.
2.13. fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Equivalent on all five to fU = fA.
2.14. fU ⊆ A ∪ fA, fA ⊆ A. Equivalent on all five to fU ⊆ A. Incorrect on all five.
2.15. A ∩ fU ⊆ fA, fA ⊆ A.

A,fA,fU FORMATS OF CARDINALITY 3
IBRT

3.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU. Incorrect on all five. Contains 1.1.
3.2. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA. Incorrect on all five. Contains 1.1.
3.3. A ∩ fA = ∅, A ∪ fU = U, A ∩ fU ⊆ fA. Incorrect on all five. Contains 1.1.
3.4. A ∩ fA = ∅, A ∪ fU = U, fA ⊆ A. Incorrect on all five. Contains 1.1.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on all five. Contains 1.1.

3.6. $A \cap fA = \emptyset$, $A \subseteq fU$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 1.1.
3.7. $A \cap fA = \emptyset$, $A \subseteq fU$, $fA \subseteq A$. Incorrect on all five. Contains 1.1.
3.8. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 1.1.
3.9. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$, $fA \subseteq A$. Incorrect on all five. Contains 1.1.
3.10. $A \cap fA = \emptyset$, $A \cap fU \subseteq fA$, $fA \subseteq A$. Incorrect on all five. Contains 1.1.
3.11. $A \cup fU = U$, $A \subseteq fU$, $fU \subseteq A \cup fA$. Incorrect on all five. Contains 2.7.
3.12. $A \cup fU = U$, $A \subseteq fU$, $A \cap fU \subseteq fA$. Equivalent on all five to $fU = U$, $A \subseteq fA$.
3.13. $A \cup fU = U$, $A \subseteq fU$, $fA \subseteq A$. Equivalent on all five to $fU = U$, $fA \subseteq A$.
3.14. $A \cup fU = U$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 2.7.
3.15. $A \cup fU = U$, $fU \subseteq A \cup fA$, $fA \subseteq A$. Incorrect on all five. Contains 2.7.
3.16. $A \cup fU = U$, $A \cap fU \subseteq fA$, $fA \subseteq A$.
3.17. $A \subseteq fU$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 2.10.
3.18. $A \subseteq fU$, $fU \subseteq A \cup fA$, $fA \subseteq A$. Incorrect on all five. Contains 2.10.
3.19. $A \subseteq fU$, $A \cap fU \subseteq fA$, $fA \subseteq A$. Equivalent on all five to $fA = A$.
3.20. $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$, $fA \subseteq A$. Incorrect on all five. Contains 2.14.

A,fA,fU FORMATS OF CARDINALITY 4
IBRT

4.1. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \subseteq fU$, $fU \subseteq A \cup fA$. Incorrect on all five. Contains 1.1.
4.2. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \subseteq fU$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 1.1.
4.3. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \subseteq fU$, $fA \subseteq A$. Incorrect on all five. Contains 1.1.
4.4. $A \cap fA = \emptyset$, $A \cup fU = U$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 1.1.
4.5. $A \cap fA = \emptyset$, $A \cup fU = U$, $fU \subseteq A \cup fA$, $fA \subseteq A$. Incorrect on all five. Contains 1.1.
4.6. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \cap fU \subseteq fA$, $fA \subseteq A$. Incorrect on all five. Contains 1.1.
4.7. $A \cap fA = \emptyset$, $A \subseteq fU$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 1.1.

4.8. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
4.9. A ∩ fA = ∅, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
4.10. A ∩ fA = ∅, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
4.11. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.7.
4.12. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.7.
4.13. A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Equivalent on all five to fU = U, fA = A.
4.14. A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.7.
4.15. A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.10.

A,fA,fU FORMATS OF CARDINALITY 5
IBRT

5.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 1.1.
5.2. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
5.3. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
5.4. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
5.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.
5.6. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.7.

A,fA,fU FORMATS OF CARDINALITY 6
IBRT

6.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 1.1.

We now list all of the formats whose status has not been determined. We use any stated equivalences that hold on all five.

1.2. A ∪ fU = U.
1.3. A ⊆ fU.
1.4. fU ⊆ A ∪ fA.
1.5. A ∩ fU ⊆ fA.

1.6. $fA \subseteq A$.
2.6. $fU = U$.
2.8. $A \cup fU = U$, $A \cap fU \subseteq fA$.
2.9. $A \cup fU = U$, $fA \subseteq A$.
2.11. $A \subseteq fA$.
2.12. $A \subseteq fU$, $fA \subseteq A$.
2.13. $fU = fA$.
2.15. $A \cap fU \subseteq fA$, $fA \subseteq A$.
3.12. $fU = U$, $A \subseteq fA$.
3.13. $fU = U$, $fA \subseteq A$.
3.16. $A \cup fU = U$, $A \cap fU \subseteq fA$, $fA \subseteq A$.
3.19. $fA = A$.
4.13. $fU = U$, $fA = A$.

We now determine the status of the above formats on the five settings.

IBRT in A,fA,fU on (SD,INF),(ELG ∩ SD,INF)

1.2. $A \cup fU = U$. Incorrect on both. Set $A = N\setminus\{0\}$.
1.3. $A \subseteq fU$. Incorrect on both. Set $A = N$.
1.4. $fU \subseteq A \cup fA$. Incorrect on both. Theorem 2.2.1.
1.5. $A \cap fU \subseteq fA$. Incorrect on both. Set $A = [\min(fU),\infty)$.
1.6. $fA \subseteq A$. Incorrect on both. Theorem 2.2.1.
2.6. $fU = U$. Incorrect on both.
2.8. $A \cup fU = U$, $A \cap fU \subseteq fA$. Incorrect on both. Contains 1.2.
2.9. $A \cup fU = U$, $fA \subseteq A$. Incorrect on both. Contains 1.6.
2.11. $A \subseteq fA$. Incorrect on both. Set $A = N$.
2.12. $A \subseteq fU$, $fA \subseteq A$. Incorrect on both. Contains 1.3.
2.13. $fU = fA$. Incorrect on both. See 1.4.
2.15. $A \cap fU \subseteq fA$, $fA \subseteq A$. Incorrect on both. Contains 1.6.
3.12. $fU = U$, $A \subseteq fA$. Incorrect on both. Contains 2.6.
3.13. $fU = U$, $fA \subseteq A$. Incorrect on both. Contains 2.6.
3.16. $A \cup fU = U$, $A \cap fU \subseteq fA$, $fA \subseteq A$. Incorrect on both. Contains 1.6.
3.19. $fA = A$. Incorrect on both. See 1.6.
4.13. $fU = U$, $fA = A$. Incorrect on both. Contains 2.6.

IBRT in A,fA,fU on (ELG,INF),(EVSD,INF)

1.2. $A \cup fU = U$. Correct on both. Theorem 2.3.3.
1.3. $A \subseteq fU$. Correct on both. Theorem 2.3.3.
1.4. $fU \subseteq A \cup fA$. Incorrect on both. Theorem 2.2.1.
1.5. $A \cap fU \subseteq fA$. Incorrect on both. Set $A = [n,\infty)$, where n is a sufficiently large element of fU.

1.6. fA ⊆ A. Incorrect on both. Theorem 2.2.1.
2.6. fU = U. Correct on both. Theorem 2.3.3.
2.8. A ∪ fU = U, A ∩ fU ⊆ fA. Incorrect on both. Contains 1.5.
2.9. A ∪ fU = U, fA ⊆ A. Incorrect on both. Contains 1.6.
2.11. A ⊆ fA. Incorrect on both. Theorem 2.2.1.
2.12. A ⊆ fU, fA ⊆ A. Incorrect on both. Contains 1.6.
2.13. fU = fA. Incorrect on both. Use Theorem 2.2.1 with D = fN. Obtain infinite A where fN ¬⊆ A ∪ fA.
2.15. A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on both. Contains 1.6.
3.12. fU = U, A ⊆ fA. Incorrect for both. Contains 2.11.
3.13. fU = U, fA ⊆ A. Incorrect for both. Contains 1.6.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect for both. Contains 1.6.
3.19. fA = A. Incorrect for both. See 1.6.
4.13. fU = U, fA = A. Incorrect for both. See 1.6.

Note the difference between (SD,INF) and (ELG,INF). E.g., 1.3 is incorrect on (SD,INF) but correct on (ELG,INF).

IBRT in A,fA,fU on (MF,INF)

1.2. A ∪ fU = U. Correct. Set f(x) = x.
1.3. A ⊆ fU. Correct. Set f(x) = x.
1.4. fU ⊆ A ∪ fA. Correct. Set f(x) = 0.
1.5. A ∩ fU ⊆ fA. Correct. Set f(x) = x.
1.6. fA ⊆ A. Correct. Set f(x) = x.
2.6. fU = U. Correct. Set f(x) = x.
2.8. A ∪ fU = U, A ∩ fU ⊆ fA. Correct. Set f(x) = x.
2.9. A ∪ fU = U, fA ⊆ A. Correct. Set f(x) = x.
2.11. A ⊆ fA. Correct. Set f(x) = x.
2.12. A ⊆ fU, fA ⊆ A. Correct. Set f(x) = x.
2.13. fU = fA. Correct. Set f(x) = 0.
2.15. A ∩ fU ⊆ fA, fA ⊆ A. Correct. Set f(x) = x.
3.12. fU = U, A ⊆ fA. Correct. Set f(x) = x.
3.13. fU = U, fA ⊆ A. Correct. Set f(x) = x.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Correct. Set f(x) = x.
3.19. fA = A. Correct. Set f(x) = x.
4.13. fU = U, fA = A. Correct. Set f(x) = x.

THEOREM 2.3.5. For IBRT in A,fA,fU on (SD,INF) and (ELG ∩ SD,INF), the only correct format is ∅. This is not true of IBRT in A,fA on (ELG,INF),(EVSD,INF),(MF,INF). IBRT in A,fA,fU on (ELG,INF) and (EVSD,INF) have the same correct formats. IBRT in A,fA,fU on (ELG,INF) and on (MF,INF) have different correct formats. IBRT in A,fA,fU on each of (SD,INF),(ELG ∩ SD,INF),(ELG,INF),(EVSD,INF) is RCA_0 secure. IBRT in A,fA,fU on (MF,INF) is ACA' secure, but not ACA_0 secure. Every correct format in A,fA,fU on (MF,INF) is RCA_0 correct. We can replace ACA' here by RCA_0 + Thin Set Theorem.

Proof: The first four claims are immediate from the given tabular classifications. For the fifth claim, it suffices to verify that the incorrectness of formats 2.7,2.10,3.11,3.14,3.15,3.17,3.18, 4.11,4.12,4.14,4.15,5.6 on these four settings is provable in RCA_0. These are the places where we have used Thin Set Theorem. In fact, it suffices to show incorrectness of 2.7 and 2.10 only, within RCA_0. But 2.7 and 2.10 each contain 1.4, which was shown to be incorrect in all four settings by Theorem 2.2.1. IBRT in A,fA,fU on (MF,INF) is ACA' secure since we only use the Thin Set Theorem (variant), which is provable in ACA'. Note that all arguments for IBRT correctness in these settings are very explicit, easily conducted in RCA_0.The last claim is by Theorem 2.2.3. QED

THEOREM 2.3.6. Let $k \geq 2$. EBRT in A,fA,fU on SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], MF[k], and IBRT in A,fA,fU on SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], are RCA_0 secure. IBRT in A,fA,fU on MF[k] is ACA_0 secure. EBRT and IBRT in A,fA on SD[k], (ELG ∩ SD)[k], ELG[k], EVSD[k], MF[k] have the same correct formats as EBRT and IBRT in SD, ELG ∩ SD, ELG, EVSD, MF, respectively.

Proof: An examination of the arguments immediately reveals that all of the incorrectness determinations given for EBRT, and all of the correctness determinations given for IBRT, involve unary and binary functions only. We can obviously pad these unary functions as k-ary functions. It is clear that the Thin Set Theorem (variant) for any fixed $k \geq 1$ is provable in ACA_0, since it relies on Ramsey’s theorem for a fixed exponent. QED

We now classify EBRT and IBRT in A,fA,fU on (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF). Much of the work is the same, but there are substantial differences that are embodied in the following results.

THEOREM 2.3.7. Let f ∈ ELG[1]. Then N\fN is infinite.

Proof: Let c be a real constant > 1. Let t ≥ 1 be such that for all n ≥ t, f(n) ≥ cn. We show that N\fN is infinite. Note that we are using only the lower bound provided by membership in ELG.

Let r ≥ 0 and s > (r+t+1)/(1 - 1/c). Then f^{-1}[r,s] ⊆ [0,s/c] ∪ [0,t]. Hence f^{-1}[r,s] has at most s/c + 1+t+1 = t+2 + s/c elements. Hence f assumes at most t+2 + s/c values in [r,s]. But by elementary algebra, t+2 + s/c < s-r+1. Hence f must assume fewer than s-r+1 values in [r,s]. Hence f omits a value in [r,s]. Since r is arbitrary and s can be taken to be a function of r (t,c are fixed), we see that f omits infinitely many values. QED

Contrast Theorem 2.3.7 with Theorem 2.3.3.

LEMMA 2.3.8. No element of EVSD[1] is surjective.

Proof: Let f ∈ EVSD. Let n be such that f is strictly dominating on [n,∞). Then f^{-1}[0,n] ⊆ [0,n-1]. By counting, there exists 0 ≤ i ≤ n such that i ∉ fN. QED

Contrast Lemma 2.3.8 with Theorem 2.3.3.

SETTINGS: (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF).

A,fA,fU FORMAT OF CARDINALITY 0
EBRT

The empty format is obviously correct, on all five.

A,fA,fU FORMATS OF CARDINALITY 1
EBRT

1.1. A ∩ fA = ∅.
1.2. A ∪ fU = U. Correct on all five. Set A = N.
1.3. A ⊆ fU.
1.4. fU ⊆ A ∪ fA. Correct on all five. Set A = N.
1.5. A ∩ fU ⊆ fA. Correct on all five. Set A = N.
1.6. fA ⊆ A. Correct on all five. Set A = N.

A,fA,fU FORMATS OF CARDINALITY 2
EBRT

2.1. A ∩ fA = ∅, A ∪ fU = U.
2.2. A ∩ fA = ∅, A ⊆ fU.

2.3. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$.
2.4. $A \cap fA = \emptyset$, $A \cap fU \subseteq fA$. Equivalent on all five to $A \cap fU = \emptyset$.
2.5. $A \cap fA = \emptyset$, $fA \subseteq A$. Equivalent on all five to $fA = \emptyset$. Incorrect on all five. Use any f.
2.6. $A \cup fU = U$, $A \subseteq fU$. Equivalent on all five to $fU = U$. Incorrect on all five. Set $rng(f) \neq N$.
2.7. $A \cup fU = U$, $fU \subseteq A \cup fA$. Correct on all five. Set $A = N$.
2.8. $A \cup fU = U$, $A \cap fU \subseteq fA$. Correct on all five. Set $A = N$.
2.9. $A \cup fU = U$, $fA \subseteq A$. Correct on all five. Set $A = N$.
2.10. $A \subseteq fU$, $fU \subseteq A \cup fA$.
2.11. $A \subseteq fU$, $A \cap fU \subseteq fA$. Equivalent on all five to $A \subseteq fA$. Incorrect on all five. Set $f(x) = 2x+1$.
2.12. $A \subseteq fU$, $fA \subseteq A$.
2.13. $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Correct on all five. Set $A = N$.
2.14. $fU \subseteq A \cup fA$, $fA \subseteq A$. Correct on all five. Set $A = N$.
2.15. $A \cap fU \subseteq fA$, $fA \subseteq A$. Correct on all five. Set $A = N$.

A,fA,fU FORMATS OF CARDINALITY 3
EBRT

3.1. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \subseteq fU$. Incorrect on all five. Contains 2.6.
3.2. $A \cap fA = \emptyset$, $A \cup fU = U$, $fU \subseteq A \cup fA$. Equivalent to $A \cap fA = \emptyset$, $A \cup fA = U$ on all five.
3.3. $A \cap fA = \emptyset$, $A \cup fU = U$, $A \cap fU \subseteq fA$. Equivalent to $A \cap fU = \emptyset$, $A \cup fU = U$ on all five. Equivalent to $A = U\setminus fU$ on all five.
3.4. $A \cap fA = \emptyset$, $A \cup fU = U$, $fA \subseteq A$. Incorrect on all five. Contains 2.5.
3.5. $A \cap fA = \emptyset$, $A \subseteq fU$, $fU \subseteq A \cup fA$.
3.6. $A \cap fA = \emptyset$, $A \subseteq fU$, $A \cap fU \subseteq fA$. Incorrect on all five. Contains 2.11.
3.7. $A \cap fA = \emptyset$, $A \subseteq fU$, $fA \subseteq A$. Incorrect on all five. Contains 2.5.
3.8. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$, $A \cap fU \subseteq fA$. Equivalent to $A \cap fA = \emptyset$, $fU \subseteq fA$, $A \cap fU = \emptyset$. Incorrect on all five. Set $f(x) = 2x+1$.
3.9. $A \cap fA = \emptyset$, $fU \subseteq A \cup fA$, $fA \subseteq A$. Incorrect on all five. Contains 2.5.
3.10. $A \cap fA = \emptyset$, $A \cap fU \subseteq fA$, $fA \subseteq A$. Incorrect on all five. Contains 2.5.

3.11. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on all five. Contains 2.6.
3.12. A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
3.13. A ∪ fU = U, A ⊆ fU, fA ⊆ A. Incorrect on all five. Contains 2.6.
3.14. A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Correct on all five. Set A = N.
3.15. A ∪ fU = U, fU ⊆ A ∪ fA, fA ⊆ A. Correct on all five. Set A = N.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Correct on all five. Set A = N.
3.17. A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.11.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A.
3.19. A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.11.
3.20. fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Correct on all five. Set A = N.

A,fA,fU FORMATS OF CARDINALITY 4
EBRT

4.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on all five. Contains 2.6.
4.2. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
4.3. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.4. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Equivalent to A = U\fU on all five. Same as 3.3 on all five.
4.5. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.6. A ∩ fA = ∅, A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.7. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.11.
4.8. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.9. A ∩ fA = ∅, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.10. A ∩ fA = ∅, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
4.11. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.

4.12. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.6.
4.13. A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.6.
4.14. A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Correct on all five. Set A = N.
4.15. A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.11.

A,fA,fU FORMATS OF CARDINALITY 5
EBRT

5.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA. Incorrect on all five. Contains 2.6.
5.2. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.3. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.4. A ∩ fA = ∅, A ∪ fU = U, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.
5.6. A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.6.

A,fA,fU FORMATS OF CARDINALITY 6
EBRT

6.1. A ∩ fA = ∅, A ∪ fU = U, A ⊆ fU, fU ⊆ A ∪ fA, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on all five. Contains 2.5.

We now list all of the formats whose status has not been determined. We use any stated equivalences that hold on all five.

1.1. A ∩ fA = ∅.
1.3. A ⊆ fU.
2.1. A ∩ fA = ∅, A ∪ fU = U.
2.2. A ∩ fA = ∅, A ⊆ fU.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA.
2.4. A ∩ fU = ∅.
2.12. A ⊆ fU, fA ⊆ A.
3.2. A ∩ fA = ∅, A ∪ fA = U.
3.3. A = U\fU.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A.

We now settle the status of each of these formats on the various settings.

EBRT in A,fA,fU on (SD[1],INF),(ELG[1] ∩ SD[1],INF)

1.1. A ∩ fA = ∅. Correct on both. Theorem 2.2.1.
1.3. A ⊆ fU. Correct on both. Set A = fN.
2.1. A ∩ fA = ∅, A ∪ fU = U. Correct on both. Complementation Theorem.
2.2. A ∩ fA = ∅, A ⊆ fU. Correct on both. Theorem 2.2.1.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Correct on both. Complementation Theorem.
2.4. A ∩ fU = ∅. Incorrect on (SD[1],INF). Set f(x) = x+1. Correct on (ELG[1] ∩ SD[1],INF). Theorem 2.3.7.
2.12. A ⊆ fU, fA ⊆ A. Correct on both. Set A = fN.
3.2. A ∩ fA = ∅, A ∪ fA = U. Correct on both. Complementation Theorem.
3.3. A = U\fU. Incorrect on (SD[1],INF). Set f(x) = x+1. Correct on (ELG[1] ∩ SD[1],INF). Theorem 2.3.7.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Correct on both. Theorem 2.3.1 with B = fN.

EBRT in A,fA,fU on (ELG[1],INF),(EVSD[1],INF)

1.1. A ∩ fA = ∅. Correct on both. Theorem 2.2.1.
1.3. A ⊆ fU. Correct on both. Set A = fN.
2.1. A ∩ fA = ∅, A ∪ fU = U. Correct on both. Theorem 2.3.2.
2.2. A ∩ fA = ∅, A ⊆ fU. Correct on both. Theorem 2.2.1.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Incorrect on both. Set f(x) = 2x.
2.4. A ∩ fU = ∅. Incorrect on (EVSD[1],INF). Set f(x) = x+1. Correct on (ELG[1],INF). Theorem 2.3.7.
3.2. A ∩ fA = ∅, A ∪ fA = U. Incorrect on both. Set f(x) = 2x.
3.3. A = N\fU. Incorrect on (EVSD[1],INF). Set f(x) = x+1. Correct on (ELG[1],INF). Theorem 2.3.7.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect on both. Set f(x) = 2x.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Correct on both. Set A = fN.

EBRT in A,fA,fU on (MF[1],INF)

1.1. A ∩ fA = ∅. Incorrect. Set f(x) = x.
1.3. A ⊆ fU. Incorrect. Set f(x) = 0.
2.1. A ∩ fA = ∅, A ∪ fU = U. Incorrect. Set f(x) = x.

2.2. A ∩ fA = ∅, A ⊆ fU. Incorrect. Set f(x) = x.
2.3. A ∩ fA = ∅, fU ⊆ A ∪ fA. Incorrect. Set f(x) = x.
2.4. A ∩ fU = ∅. Incorrect. Set f(x) = x.
2.12. A ⊆ fU, fA ⊆ A. Incorrect. Set f(x) = 0.
3.2. A ∩ fA = ∅, A ∪ fA = U. Incorrect. Set f(x) = x.
3.3. A = N\fU. Incorrect. Set f(x) = x.
3.5. A ∩ fA = ∅, A ⊆ fU, fU ⊆ A ∪ fA. Incorrect. Set f(x) = x.
3.18. A ⊆ fU, fU ⊆ A ∪ fA, fA ⊆ A. Incorrect. Set f(x) = 0.

THEOREM 2.3.9. EBRT in A,fA,fU on the ten BRT settings (SD,INF), (ELG ∩ SD,INF), (ELG,INF), (EVSD,INF), (MF,INF). (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF), are RCA_0 secure. They also have different correct formats, with the following exceptions. (SD,INF), (ELG ∩ SD,INF), (SD[1],INF) have the same; (ELG,INF), (EVSD,INF), (EVSD[1],INF) have the same; (MF,INF), (MF[1],INF) have the same. In particular, (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF), all differ on EBRT in A,fA,fU.

Proof: Our entire analysis of EBRT in this section takes place in RCA_0. To compare the multivariate settings with the unary settings, we have only to examine where we use a function that is not unary for an incorrectness determination in the multivariate setting.

(SD,INF), (SD[1],INF). In 2.4, 3.3, 4.4, we use Theorem 2.3.3, which involves functions that are not unary. However, we can instead use f(x) = x+1, which lies in SD[1].

(EVSD,INF), (EVSD[1],INF). In 2.4, 3.3, 4.4, we use Theorem 2.3.3, which involves functions that are not unary. However, we can instead use f(x) = x+1, which lies in EVSD[1].

(MF,INF), (MF[1],INF). In 2.4, 3.3, 4.4, we use Theorem 2.3.3, which involves functions that are not unary. However, we can instead use f(x) = x+1, which lies in MF[1].

It suffices to verify that EBRT in A,fA,fU pairwise differ on

(SD,INF).
(ELG,INF).

(MF,INF).
(ELG[1] ∩ SD[1],INF).
(ELG[1],INF)

(ELG[1] ∩ SD[1],INF) and (ELG[1],INF) both differ from (SD,INF), (ELG,INF), (MF,INF) at 2.4. (ELG[1] ∩ SD[1],INF) and (ELG[1],INF) differ at 2.3. From Theorem 2.3.4, we know that (SD,INF), (ELG,INF), (MF,INF) differ. QED

We now come to IBRT in the five unary settings. First note that in the earlier table of formats of cardinalities 0-6 on IBRT in A,fA,fU, compiled earlier, the only determinations were of incorrectness. Obviously those determinations still apply. So we can jump ahead to where we list the formats that remain undetermined:

1.2. A ∪ fU = U.
1.3. A ⊆ fU.
1.4. fU ⊆ A ∪ fA.
1.5. A ∩ fU ⊆ fA.
1.6. fA ⊆ A.
2.6. fU = U.
2.8. A ∪ fU = U, A ∩ fU ⊆ fA.
2.9. A ∪ fU = U, fA ⊆ A.
2.11. A ⊆ fA.
2.12. A ⊆ fU, fA ⊆ A.
2.13. fU = fA.
2.15. A ∩ fU ⊆ fA, fA ⊆ A.
3.12. fU = U, A ⊆ fA.
3.13. fU = U, fA ⊆ A.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A.
3.19. fA = A.
4.13. fU = U, fA = A.

We now determine the status of the above formats on the five unary settings.

IBRT in A,fA,fU on (SD[1],INF),(ELG[1] ∩ SD[1],INF)

Since the only correct format for IBRT in A,fA,fU on (SD[1],INF), (ELG[1] ∩ SD[1],INF) is ∅, the only correct format for IBRT in A,fA,fU on (SD,INF), (ELG ∩ SD,INF) is ∅.

IBRT in A,fA,fU on (ELG[1],INF),(EVSD[1],INF)

1.2. A ∪ fU = U. Incorrect on both. Lemma 2.3.8.

1.3. A ⊆ fU. Incorrect on both. Lemma 2.3.8.
1.4. fU ⊆ A ∪ fA. Incorrect on both. Theorem 2.2.1.
1.5. A ∩ fU ⊆ fA. Incorrect on both. FIX!!! Use Theorem 2.2.1 with D = fN. Obtain infinite A disjoint from fA, where fN ¬⊆ A ∪ fA. If A ∩ fN ⊆ fA then fN ⊆ fA.
1.6. fA ⊆ A. Incorrect on both. Theorem 2.2.1.
2.6. fU = U. Incorrect on both. Lemma 2.3.8.
2.8. A ∪ fU = U, A ∩ fU ⊆ fA. Incorrect on both. Contains 1.5.
2.9. A ∪ fU = U, fA ⊆ A. Incorrect on both. Contains 1.6.
2.11. A ⊆ fA. Incorrect on both. Theorem 2.2.1.
2.12. A ⊆ fU, fA ⊆ A. Incorrect on both. Contains 1.6.
2.13. fU = fA. Incorrect on both. Use Theorem 2.2.1 with D = fU.
2.15. A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on both. Contains 1.6.
3.12. fU = U, A ⊆ fA. Incorrect on both. Contains 2.11.
3.13. fU = U, fA ⊆ A. Incorrect on both. Contains 1.6.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Incorrect on both. Contains 1.6.
3.19. fA = A. Incorrect on both. See 1.6.
4.13. fU = U, fA = A. Incorrect on both. See 1.6.

We now see that in IBRT on SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), every format is incorrect.

IBRT in A,fA,fU on (MF[1],INF)

1.2. A ∪ fU = U. Correct. Set f(x) = x.
1.3. A ⊆ fU. Correct. Set f(x) = x.
1.4. fU ⊆ A ∪ fA. Correct. Set f(x) = 0.
1.5. A ∩ fU ⊆ fA. Correct. Set f(x) = x.
1.6. fA ⊆ A. Correct. Set f(x) = x.
2.6. fU = U. Correct. Set f(x) = x.
2.8. A ∪ fU = U, A ∩ fU ⊆ fA. Correct. Set f(x) = x.
2.9. A ∪ fU = U, fA ⊆ A. Correct. Set f(x) = x.
2.11. A ⊆ fA. Correct. Set f(x) = x.
2.12. A ⊆ fU, fA ⊆ A. Correct. Set f(x) = x.
2.13. fU = fA. Correct. Set f(x) = 0.
2.15. A ∩ fU ⊆ fA, fA ⊆ A. Correct. Set f(x) = x.
3.12. fU = U, A ⊆ fA. Correct. Set f(x) = x.
3.13. fU = U, fA ⊆ A. Correct. Set f(x) = x.
3.16. A ∪ fU = U, A ∩ fU ⊆ fA, fA ⊆ A. Correct. Set f(x) = x.
3.19. fA = A. Correct. Set f(x) = x.
4.13. fU = U, fA = A. Correct. Set f(x) = x.

THEOREM 2.3.10. IBRT in A,fA,fU on (SD,INF), (ELG ∩ SD,INF), (SD[1],INF),(ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (SD[1],INF), (ELG[1] ∩ SD[1],INF) have only the correct format ∅. IBRT in A,fA,fU on (MF,INF) and (MF[1],INF) have the same correct formats. IBRT in A,fA,fU on (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF) are RCA_0 secure.

Proof: By inspection. Also, Thin Set Theorem (variant) is provable in RCA_0 by Lemma 2.2.4 QED

Note that by Theorem 2.3.10, there are exactly five different behaviors of the ten BRT settings (SD,INF), (ELG ∩ SD,INF), (ELG,INF), (EVSD,INF), (MF,INF). (SD[1],INF), (ELG[1] ∩ SD[1],INF), (ELG[1],INF), (EVSD[1],INF), (MF[1],INF) under EBRT in A,fA,fU. By Theorem 2.3.10, there are three under IBRT in A,fA,fU.