3.10. ABAC.

Recall the reduced AB table from section 3.5.

REDUCED AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

The reduced AC table is obtained from the reduced AB table by interchanging B and C. We use 1'-6' to avoid confusion.

REDUCED AC

1′. A ∪. fA ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
2′. A ∪. fA ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
3′. A ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
4′. B ∪. fA ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
5′. B ∪. fA ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
6′. B ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.

We will use the reduced AB table and the reduced AC table.

Note that each i,j′ is equivalent to j,i′, because each i,j′ is sent to i′,j by interchanging B and C.

Hence we need only consider i,j′ where i ≤ j′.

We need to determine the status of INF, AL, ALF, FIN, NON for each pair.

1,1′. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
1,2′. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gC. INF. AL. ALF. FIN, NON.
1,3′. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
1,4′. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,5′. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,6′. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

2,2′. A ∪. fA ⊆ B ∪. gB. A ∪. fA ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
2,3′. A ∪. fA ⊆ B ∪. gB. A ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
2,4′. A ∪. fA ⊆ B ∪. gB. B ∪. fA ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,5′. A ∪. fA ⊆ B ∪. gB. B ∪. fA ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,6′. A ∪. fA ⊆ B ∪. gB. B ∪. fA ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,3′. A ∪. fA ⊆ B ∪. gC. A ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
3,4′. A ∪. fA ⊆ B ∪. gC. B ∪. fA ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,5′. A ∪. fA ⊆ B ∪. gC. B ∪. fA ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,6′. A ∪. fA ⊆ B ∪. gC. B ∪. fA ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,4′. C ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,5′. C ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,6′. C ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,5′. C ∪. fA ⊆ B ∪. gB. B ∪. fA ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,6′. C ∪. fA ⊆ B ∪. gB. B ∪. fA ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FiN. ¬NON.
6,6′. C ∪. fA ⊆ B ∪. gC. B ∪. fA ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

LEMMA 3.10.1. fA ⊆ B ∪. gY, B ∩ fA = ∅ has ¬NON.

Proof: Define f,g ∈ ELG as follows. f(n) = 2n+2, g(n) = 2n+1. Let X ∪. fA ⊆ B ∪. gY, B ∪. fA ⊆ Z ∪. gW, where A,B,C are nonempty.

Clearly fA ⊆ B. This contradicts B ∩ fA = ∅. QED

LEMMA 3.10.2. 1,4′ - 1,6′, 2,4′ - 2,6′, 3,4′ - 6,6′ have ¬NON.

Proof: By Lemma 3.10.1. QED

LEMMA 3.10.3. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gX has INF, ALF, provided X ∈ {A,B}, even for EVSD.

Proof: Let f,g ∈ EVSD. By Lemma 3.2.5, let A ⊆ N be infinite, where A is disjoint from fA ∪ g(A ∪ fA), and min(A) is sufficiently large.

Let B = (A ∪ fA)\gA. Let C = (A ∪ fA)\gX.

Clearly A ∩ fA = B ∩ gA = A ∩ gA = A ∩ gB = C ∩ gX = ∅. Hence A ⊆ B, A ⊆ C. Also A ∪ fA ⊆ B ∪ gA, A ∪ fA ⊆ C ∪ gX. This establishes INF.

We can repeat the argument using A of any given finite cardinality. This establishes ALF. QED

LEMMA 3.10.4. 1,1’ and 1,3’ have INF, ALF, even for EVSD.

Proof: Immediate from Lemma 3.10.3. QED

The following pertains to 1,2’.

LEMMA 3.10.5. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gC has INF, ALF, even for EVSD.

Proof: Let f,g ∈ EVSD. By Lemma 3.2.5, let A ⊆ N be infinite, where A is disjoint from fA ∪ g(A ∪ fA), and min(A) is sufficiently large.

Let B = (A ∪ fA)\gA. By Lemma 3.3.3, let C be unique such that C ⊆ A ∪ fA ⊆ C ∪. gC. Then C is infinite.

Clearly A ∩ fA = B ∩ gA = C ∩ gC = A ∩ gA = ∅. Also A ∩ gC ⊆ A ∩ g(A ∪ fA) = ∅. Hence A ⊆ B, A ⊆ C. Also A ∪ fA ⊆ B ∪ gA, A ∪ fA ⊆ C ∪ gC. This establishes INF.

We can repeat the argument using A of any given finite cardinality. This establishes ALF. QED

The following pertains to 2,3’.

LEMMA 3.10.6. A ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ C ∪. gB has INF, ALF, even for EVSD.

Proof: Let f,g ∈ EVSD. By Lemma 3.2.5, let A ⊆ N be infinite, where A is disjoint from fA ∪ g(A ∪ fA), and min(A) is sufficiently large.

By Lemma 3.3.3, let B be unique such that B ⊆ A ∪ fA ⊆ B ∪. gB. Define C = (A ∪ fA)\gB.

Clearly A ∩ fA = B ∩ gB = C ∩ gB = ∅. Also A ∩ gB ⊆ A ∩ g(A ∪ fA) = ∅. Hence A ⊆ B, A ⊆ C. Also A ∪ fA ⊆ B ∪ gB, A ∪ fA ⊆ C ∪ gB. This establishes INF.

We can repeat the argument using A of any given finite cardinality. This establishes ALF. QED

The following pertains to 2,2'.

LEMMA 3.10.7. A ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ C ∪. gC has INF, ALF, even for EVSD.

Proof: From the AB table, A ∪. fA ⊆ B ∪. gB has INF, ALF. Replace C by B in the cited pair. QED

The following pertains to 3,3'.

LEMMA 3.10.8. A ∪. fA ⊆ B ∪. gC. A ∪. fA ⊆ C ∪. gB has INF, ALF, even for EVSD.

Proof: A ∪. fA ⊆ B ∪. gB has INF, ALF, by the AB table. Replace C by B in the cited pair. QED