3.11. ABBA.

Recall the reduced AB table from section 3.5.

REDUCED AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

Recall the reduced BA table from section 3.6.

REDUCED BA

1′. B ∪. fB ⊆ A ∪. gB. INF. AL. ALF. FIN. NON.
2′. B ∪. fB ⊆ A ∪. gA. INF. AL. ALF. FIN. NON.
3′. B ∪. fB ⊆ A ∪. gC. INF. AL. ALF. FIN. NON.
4′. C ∪. fB ⊆ A ∪. gB. INF. AL. ALF. FIN. NON.
5′. C ∪. fB ⊆ A ∪. gA. INF. AL. ALF. FIN. NON.
6′. C ∪. fB ⊆ A ∪. gC. INF. AL. ALF. FIN. NON.

This results in 36 ordered pairs.

We can take advantage of symmetry through interchanging A with B as follows. Clearly (i,j′) and (j,i′) are equivalent, since interchanging A and B takes us from p to p′ and back. So we can require that i ≤ j. Thus we have the following 21 ordered pairs to consider.

We need to determine the status of all attributes INF, Al, ALF, FIN, NON, for each pair.

1,1′. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,2′. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,3′. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,4′. A ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,5′. A ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,6′. A ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

2,2′. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,3′. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,4′. A ∪. fA ⊆ B ∪. gB, C ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,5′. A ∪. fA ⊆ B ∪. gB, C ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,6′. A ∪. fA ⊆ B ∪. gB, C ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,3′. A ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,4′. A ∪. fA ⊆ B ∪. gC, C ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,5′. A ∪. fA ⊆ B ∪. gC, C ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,6′. A ∪. fA ⊆ B ∪. gC, C ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,4′. C ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,5′. C ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,6′. C ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,5′. C ∪. fA ⊆ B ∪. gB, C ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,6′. C ∪. fA ⊆ B ∪. gB, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
6,6′. C ∪. fA ⊆ B ∪. gC, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.11.1. 1,1′ - 6,6′ have ¬INF, ¬FIN.

Proof: Let f be as given by Lemma 3.2.4. Let g ∈ ELG be defined by g(n) = 2n+1. Let

X ∪. fA ⊆ B ∪. gY
S ∪. fB ⊆ A ∪. gT

where X,A,B,Y,S,T are nonempty subsets of N. Then fA ∩ 2N ⊆ B and fB ∩ 2N ⊆ A. Hence f(fA ∩ 2N) ∩ 2N ⊆ fB ∩ 2N ⊆ A. By Lemma 3.2.4, fA is cofinite. Thus A is infinite. This establishes ¬FIN. Also X is finite, since X ∩ fA = ∅. This establishes ¬INF. QED

Lemma 3.11.1 establishes that we have ¬INF, ¬ALF, ¬FIN for all of the pairs of clauses considered in this section. It remains to determine the status of AL and NON.

LEMMA 3.11.2. $fA \subseteq B \cup. gY$, $fB \subseteq A \cup. gZ$, $A \cap fA = \emptyset$ has ¬NON.

Proof: Define $f,g \in ELG$ as follows. For all $n < m$, let $f(n,n) = 2n+2$, $f(n,m) = 2m$, $f(m,n) = 4m$, $g(n) = 2n+1$. Let $fA \subseteq B \cup. gY$, $fB \subseteq A \cup. gZ$, $A \cap fA = \emptyset$, where A,B,Y,Z are nonempty subsets of N.

Let $n \in B$. Then $2n+2 \in fB$, $2n+2 \in A$, $4n+6 \in fA$, $4n+6 \in B$. Since $n < 4n+6$ are from B, we have $8n+12 \in fB$, $8n+12 \in A$. Since $2n+2 < 8n+12$ are from A, we have $16n+24 \in fA$. Also since $n < 4n+6$ are from B, we have $16n+24 \in fB$, $16n+24 \in A$. This contradicts $A \cap fA = \emptyset$. QED

LEMMA 3.11.3. 1,1′ - 3,6′ have ¬NON.

Proof: By Lemma 3.11.2. QED

LEMMA 3.11.4. $C \cup. fA \subseteq B \cup. gX$, $C \cup. fB \subseteq A \cup. gY$ has AL.

Proof: Let $f,g \in ELG$ and $p \geq 0$. Let $C = [n,n+p]$, where n is sufficiently large. Throw all elements of $[n,n+p]$ into A,B. A,B will have no elements $< n$.

We determine membership of all $k > n+p$ in A,B by induction as follows. Suppose membership in A,B has been determined for all integers $< k$, where $k > n+p$ is fixed. If k is not already in gX then put k in B. If k is not already in gY then put k in A.

Note that $C \subseteq A,B \subseteq [n,\infty)$, $C \cap fA = C \cap fB = B \cap gX = A \cap gY = \emptyset$. Also we have $fA \subseteq [n,\infty) \subseteq B \cup gX$, $fB \subseteq [n,\infty) \subseteq A \cup gY$. QED

LEMMA 3.11.5. 4,4′ - 6,6′ have AL.

Proof: By Lemma 3.11.4. QED