3.12. ABBC.

Recall the following reduced table for AB from section 3.5.

REDUCED AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

The reduced table for BC is obtained from the reduced table for AB via the permutation that sends A to B, B to C, and C to A. We use 1'-6' to avoid confusion.

REDUCED BC

1′. B ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
2′. B ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
3′. B ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
4′. A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
5′. A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
6′. A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

This results in 36 ordered pairs, which we divide into six cases. We begin with two Lemmas.

We will determine the status of all attributes INF, AL, ALF, FIN, NON, for all ordered pairs.

LEMMA 3.12.1. C ∪. fX ⊆ B ∪. gY, Z ∪. fB ⊆ C ∪. gW has ¬INF, ¬FIN.

Proof: Let f be as given by Lemma 3.2.1. Let g ∈ ELG be given by g(n) = 2n+1. Let C ∪. fX ⊆ B ∪. gY, Z ∪. fB ⊆ C ∪. gW, where A,B,C are nonempty.

Clearly fB ∩ 2N ⊆ C. By C ⊆ B ∪ gY, we have fB ∩ 2N ⊆ B. Hence by Lemma 3.2.1, fB is cofinite. Hence B is infinite. This establishes that ¬FIN. Also Z is finite. This establishes that ¬INF. QED

LEMMA 3.12.2. C ∪. fX ⊆ B ∪. gY, Z ∪. fB ⊆ C ∪. gW, B ∩ fB = ∅ has ¬NON.

Proof: We can continue the proof of Lemma 3.12.1. Using fB is cofinite and B is finite, we obtain an immediate contradiction from B ∩ fB = ∅. QED

We use Lemmas 3.12.1 and 3.12.2 in cases 5,6 below.

part 1. A ∪. fA ⊆ B ∪. gA.

1,1′. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,2′. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,3′. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
1,4′. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
1,5′. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
1,6′. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

The following pertains to 1,4′, 1,6′.

LEMMA 3.12.3. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gX has INF, ALF provided X ∈ {A,B}, even for EVSD.

Proof: Let f,g ∈ EVSD. Let n be sufficiently large. By Lemma 3.2.5, let A ⊆ [n,∞) be infinite, where A is disjoint from f(A ∪ fA) ∪ g(A ∪ fA). Let B = (A ∪ fA)\gA, and C = (A ∪ fB)\gX.

Clearly A ∩ fA = B ∩ gA = A ∩ fB = C ∩ gX = A ∩ gA = A ∩ gB = ∅. Hence A ⊆ B and A ⊆ C. Also fA ⊆ B ∪ gA and fB ⊆ C ∪ gX. This establishes INF.

We can repeat the argument where A is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 1,5′.

LEMMA 3.12.4. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gC has INF, ALF, even for EVSD.

Proof: Let f,g ∈ EVSD. Let n be sufficiently large. By Lemma 3.2.5, let A ⊆ [n,∞) be infinite, where A is disjoint from f(A ∪ fA) ∪ g(A ∪ fA) ∪ g(A ∪ f(A ∪ fA)). Let B = (A

∪ fA)\gA. By Lemma 3.3.3, let C be unique such that C ⊆ A ∪ fB ⊆ C ∪. gC.

Clearly A ∩ fA = B ∩ gA = A ∩ fB = C ∩ gC = A ∩ gA = A ∩ gC = ∅. Hence A ⊆ B and A ⊆ C. Also fA ⊆ B ∪ gA and fB ⊆ C ∪ gC. This establishes INF.

We can repeat the proof where A is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 1,1', 1,2', 1,3'.

LEMMA 3.12.5. A ∪. fA ⊆ B ∪. gA, B ∩ fB = ∅ has ¬NON.

Proof: Define f,g ∈ ELG as follows. Let f(n) = 2n+2 and g(n) = 2n+1. Let A ∪. fA ⊆ B ∪. gA, B ∩ fB = ∅, where A,B are nonempty.

Let n = min(A). Then n ∉ gA, n ∈ B, 2n+2 ∈ fB, 2n+2 ∈ fA, 2n+2 ∈ B. This contradicts B ∩ fB = ∅. QED

part 2. A ∪. fA ⊆ B ∪. gB.

2,1'. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. FIN. NON.
2,2'. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.
2,3'. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. FIN. NON.
2,4'. A ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
2,5'. A ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
2,6'. A ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

The following pertains to 2,4', 2,6'.

LEMMA 3.12.6. A ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gX has INF, ALF, provided X ∈ {A,B}, even for EVSD.

Proof: Let f,g ∈ EVSD. Let n be sufficiently large. By Lemma 3.2.5, let A ⊆ [n,∞) be infinite, where A is disjoint from f(A ∪ fA) ∪ g(A ∪ fA). By Lemma 3.3.3, let B be unique such that B ⊆ A ∪ fA ⊆ B ∪. gB. Let C = (A ∪ fB)\gX.

Clearly $A \cap fA = B \cap gB = A \cap fB = C \cap gX = A \cap gB = A \cap gA = \emptyset$. Hence $A \subseteq B$ and $A \subseteq C$. Also $fA \subseteq B \cup gB$ and $fB \subseteq C \cup gX$. This establishes INF.

We can repeat the argument where A is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 2,5'.

LEMMA 3.12.7. $A \cup. fA \subseteq B \cup. gB$, $A \cup. fB \subseteq C \cup. gC$ has INF, ALF, even for EVSD.

Proof: Let $f,g \in$ EVSD. Let n be sufficiently large. By Lemma 3.2.5, let $A \subseteq [n,\infty)$ be infinite, where A is disjoint from $f(A \cup fA) \cup g(A \cup fA) \cup g(A \cup f(A \cup fA))$. By Lemma 3.3.3, let B be unique such that $B \subseteq A \cup fA \subseteq B \cup. gB$. By Lemma 3.3.3, let C be unique such that $C \subseteq A \cup fB \subseteq C \cup. gC$.

Clearly $A \cap fA = B \cap gB = A \cap fB = C \cap gC = A \cap gB = A \cap gC = \emptyset$. Hence $A \subseteq B$ and $A \subseteq C$. Also $fA \subseteq B \cup gB$ and $fB \subseteq C \cup gC$. This establishes INF.

We can repeat the argument where A is chosen to be of any finite cardinality. This establishes ALF. QED

The following pertains to 2,1', 2,3'.

LEMMA 3.12.8. $A \cup. fA \subseteq B \cup. gB$, $B \cup. fB \subseteq C \cup. gX$ has FIN, provided $X \in \{A,B\}$.

Proof: Let $f,g \in$ ELG. We claim that there exists arbitrarily large n such that $f(n,...,n) \neq f(g(n,...,n),...,g(n,...,n))$. Suppose this is false. I.e., let r be such that for all $n \geq r$, $f(n,...,n) = f(g(n,...,n),...,g(n,...,n))$. We can assume that r is chosen so that f,g is strictly dominating on $[r,\infty)$.

Define $t_0 = r$, $t_{i+1} = g(t_i,...,t_i)$. An obvious induction shows that $r \leq t_0 < t_1 < ...$.

We now prove by induction that for all $i \geq 0$,

$$f(r,...,r) = f(t_i,...,t_i).$$

Obviously this is true for $i = 0$. Suppose this is true for a given $i \geq 0$. Then

$$f(r,...,r) = f(t_i,...,t_i).$$
$$t_i \geq r.$$
$$f(t_i,...,t_i) = f(g(t_i,...,t_i),...,g(t_i,...,t_i)).$$
$$f(r,...,r) = f(t_{i+1},...,t_{i+1}).$$

However some t_i is greater than f(r,...,r), since the t's are strictly increasing. This is a contradiction. The claim is now established.

Now let n be sufficiently large with the property that f(n,...,n) ≠ f(g(n,...,n),...,g(n,...,n)). Let A = {g(n,...,n)}. Let B = {n,f(g(n,...,n),...,g(n,...,n))}. Let C = (B ∪ fB)\gX.

Clearly A ∩ fA = B ∩ gB = B ∩ fB = C ∩ gX = ∅. Also A ⊆ gB, fA ⊆ B, B ∪ fB ⊆ C ∪ gX. In addition, n ∉ gX, n ∈ B, and so n ∈ C. Hence A,B,C are nonempty finite sets. QED

The following pertains to 2,2'.

LEMMA 3.12.9. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gC has FIN.

Proof: Let f,g ∈ ELG. We define n,A,B exactly as in the proof of Lemma 3.12.8. By Lemma 3.3.3, let C be unique such that C ⊆ B ∪ fB ⊆ C ∪. gC.

Clearly A ∩ fA = B ∩ gB = B ∩ fB = C ∩ gC = ∅. Also A ⊆ gB, fA ⊆ B, B ∪ fB ⊆ C ∪ gC. In addition, n ∉ gC, and so n ∈ C. Hence A,B,C are nonempty finite sets. QED

The following pertains to 2,1', 2,2', 2,3'.

LEMMA 3.12.10. fA ⊆ B ∪. gX, B ∩ fB = ∅ has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(m,n) = f(n,m) = 4m+6, g(n) = 2n+1. Let fA ⊆ B ∪. gX, B ∩ fB = ∅, where A,B,C have at least two elements. Let n < m be from A. Then 2m+2,4m+6 ∈ fA, 2m+2,4m+6 ∈ B, 4m+6 ∈ fB. This contradicts B ∩ fB = ∅. QED

part 3. A ∪. fA ⊆ B ∪. gC.

3,1'. A ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. FIN. NON.

3,2′. A ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.
3,3′. A ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. FIN. NON.
3,4′. A ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
3,5′. A ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
3,6′. A ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

LEMMA 3.12.11. 3,1′ - 3,3′ have ¬AL.

Proof: By Lemma 3.12.10. QED

The following pertains to 3,1′, 3,3′.

LEMMA 3.12.12. A ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gX has FIN, where X ∈ {A,B}.

Proof: Let f,g ∈ ELG. Let n be sufficiently large. Define A = {g(n,...,n)}, B = {f(g(n,...,n),...,g(n,...,n))}, C = (B ∪ fB ∪ {n})\gX.

Obviously A ∩ fA = B ∩ fB = C ∩ gX = ∅. Also n ∉ gX, n ∈ C. Hence A ⊆ gC and fA ⊆ B. Therefore A ∪ fA ⊆ B ∪ gC. Obviously B ∪ fB ⊆ C ∪ gX.

It remains to verify that B ∩ gC = ∅. Every element of C is either n or f(g(n,...,n),...,g(n,...,n)) or the value of a term of depth ≤ 3 in f,g,n with f(g(n,...,n),...,g(n,...,n)) as a subterm. Hence every element of gC is either g(n,...,n) or the value of a term in f,g,n of depth ≤ 4 with f(g(n,...,n),...,g(n,...,n)) as a proper subterm. Since n is sufficiently large, f(g(n,...,n),...,g(n,...,n)) does not lie in gC. QED

The following pertains to 3,2′.

LEMMA 3.12.13. A ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gC has FIN.

Proof: Let f,g ∈ ELG. Let n be sufficiently large. Define A = {g(n,...,n)}, B = {f(g(n,...,n),...,g(n,...,n))}. By Lemma 3.3.3, let C be unique such that C ⊆ B ∪ fB ∪ {n} ⊆ C ∪. gC.

Obviously A ∩ fA = B ∩ fB = C ∩ gC = ∅. Also n ∉ gC, n ∈ C. A ⊆ gC, and fA ⊆ B. Therefore A ∪ fA ⊆ B ∪ gC. In addition, B ∪ fB ⊆ C ∪ gC.

It remains to verify that B ∩ gC = ∅. Argue exactly as in the proof of Lemma 3.12.12. QED

The following pertains to 3,4′, 3,5′, 3,6′.

LEMMA 3.12.14. A ∪. fA ⊆ B ∪. gC. A ∪. fB ⊆ C ∪. gX has INF, ALF, even for EVSD.

Proof: Let f,g ∈ EVSD. Let n be sufficiently large. By Lemma 3.2.5, let A ⊆ [n,∞) be infinite, where A is disjoint from f(A ∪ fA) ∪ g(A ∪ f(A ∪ fA)). We inductively determine membership in B,C for all elements of [n,∞). B,C will have no elements < n.

Suppose membership in B,C has been determined for all elements of [n,k), k ≥ n. We now determine membership in B,C for k. If k is already in A ∪ fA and k is not yet in gC, put k ∈ B. If k is already in A ∪ fB and k is not yet in gX, put k in C.

Clearly B ⊆ A ∪ fA and C ⊆ A ∪ fB ⊆ A ∪ f(A ∪ fA). Hence A ∩ fA = A ∩ fB = C ∩ gX = ∅. Also A ∪ fA ⊆ B ∪ gC and A ∪ fB ⊆ C ∪ gX. In addition, A ∩ gC ⊆ A ∩ g(A ∪ fB) ⊆ A ∩ g(A ∪ f(A ∪ fA)) = ∅, and so A ∩ gX = ∅. Hence A ⊆ B, A ⊆ C. This establishes INF.

We can instead use A of any finite cardinality. We obtain finite B,C with A ⊆ B,C. This establishes ALF. QED

part 4. C ∪. fA ⊆ B ∪. gA.

4,1′. C ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,2′. C ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,3′. C ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,4′. C ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,5′. C ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4,6′. C ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

The following pertains to 4,1′, 4,2′, 4,3′.

LEMMA 3.12.15. C ∪. fA ⊆ B ∪. gA. B ∪. fB ⊆ C ∪. gX has ¬NON.

Proof: Let f be as given by Lemma 3.2.1. Define g ∈ ELG by g(n) = 2n+1. Let C ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ C ∪. gX, where A,B,C are nonempty.

Let n ∈ fB ∩ 2N. Then n ∈ C, n ∈ B. Hence fB ∩ 2N ⊆ B. By Lemma 3.2.1, fB is cofinite. Hence B is infinite. This contradicts B ∩ fB = ∅. QED

The following pertains to 4,4′.

LEMMA 3.12.16. C ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gB has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(2n,2n,2n) = f(2n+1,2n+1,2n+1) = 4n, f(n,m,m) = 2m, f(n,m,n) = 4m, f(m,n,n) = 8m, g(2n) = g(2n+1) = 4n+1. For all other triples a,b,c, let f(a,b,c) = 2|a,b,c|.

We claim that

f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(g(m),g(m),g(m)).

To see this, let m = 2r ∨ m = 2r+1. Then

f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(4r,4r,4r) = 8r

and

f(g(m),g(m),g(m)) = f(4r+1,4r+1,4r+1) = 8r.

Now let C ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ C ∪. gB, where A,B,C are nonempty. Let n ∈ A. Then n ∈ C ∪. gB.

case 1. n ∈ C. Then n ∈ B ∪ gA. First suppose n ∈ B. Then f(n,n,n) ∈ C ∪ gB. Hence f(n,n,n) ∈ C. This contradicts C ∩ fA = ∅.

Now suppose n ∈ gA. Let n = g(m), m ∈ A, m < n. Then 2n-2,4n-4,8n-8 ∈ fA, and so 2n-2,4n-4,8n-8 ∈ B, 8n-8 ∈ fB, 8n-8 ∈ C. This contradicts C ∩ fA = ∅.

case 2. $n \in gB$. Let $n = g(m)$, $m \in B$. Then $f(m,m,m) \in fB$, $f(m,m,m) \in C$. Hence $f(m,m,m) \in B$. Therefore $f(f(m,m,m),f(m,m,m),f(m,m,m)) \in fB$, $f(f(m,m,m),f(m,m,m),f(m,m,m)) \in C$. Note that $f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(g(m),g(m),g(m)) = f(n,n,n) \in fA$. This contradicts $C \cap fA = \emptyset$.

QED

The following pertains to 4,6'.

LEMMA 3.12.17. $C \cup. fA \subseteq B \cup. gA$, $A \cup. fB \subseteq C \cup. gA$ has ¬NON.

Proof: Define f,g as in the proof of Lemma 3.12.16. Now let $C \cup. fA \subseteq B \cup. gA$, $A \cup. fB \subseteq C \cup. gA$, where A,B,C are nonempty. Let $n \in A$. Then $n \in C \cup. gA$.

case 1. $n \in gA$. Let $n = g(m)$, $m \in A$, $m < n$. Then $2n-2,4n-4,8n-8 \in fA$, $2n-2,4n-4,8n-8 \in B$, $8n-8 \in fB$, $8n-8 \in C$. This contradicts $C \cap fA = \emptyset$.

case 2. $n \in C$. Then $n \notin gA$, $n \in B$, $f(n,n,n) \in fB$, $f(n,n,n) \in C$. Since $f(n,n,n) \in fA$, this contradicts $C \cap fA = \emptyset$.

QED

The following pertains to 4,5'.

LEMMA 3.12.18. $C \cup. fA \subseteq B \cup. gA$, $A \cup. fB \subseteq C \cup. gC$ has ¬NON.

Proof: Define f,g as in the proof of Lemma 3.12.16. Now let $C \cup. fA \subseteq B \cup. gA$, $A \cup. fB \subseteq C \cup. gC$, where A,B,C are nonempty. Let $n = \min(A)$. Then $n \in C \cup. gC$.

case 1. $n \in C$. By the choice of n, $n \notin gA$, $n \in B$. Hence $f(n,n,n) \in fB$, $f(n,n,n) \in C$. Since $f(n,n,n) \in fA$, this contradicts $C \cap fA = \emptyset$.

case 2. $n \in gC$. Let $n = g(m)$, $m \in C$, $m < n$. Then $m \in B \cup gA$. By the choice of n, $m \notin gA$, $m \in B$. Hence $f(m,m,m) \in fB$, $f(m,m,m) \in C$, $f(m,m,m) \in B \cup gA$.

We claim that $f(m,m,m) \notin gA$. To see this, note that by quantitative considerations, $f(m,m,m) \in gA$ implies that there is an element of A that is ≤ m < n, which contradicts the choice of n.

Hence f(m,m,m) ∈ B. Therefore

$$f(f(m,m,m),f(m,m,m),f(m,m,m)) \in fB.$$
$$f(f(m,m,m),f(m,m,m),f(m,m,m)) \in C.$$

As in the proof of Lemma 3.12.16,

$$f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(g(m),g(m),g(m)) = f(n,n,n) \in fA.$$

This contradicts C ∩ fA = ∅.

QED

part 5. C ∪. fA ⊆ B ∪. gB.

5,1′. C ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,2′. C ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,3′. C ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,4′. C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,5′. C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,6′. C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

LEMMA 3.12.19. 5,1′, 5,2′, 5,3′ have ¬NON.

Proof: By Lemma 3.12.2. QED

The following pertains to 5,4′.

LEMMA 3.12.20. C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gB has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n) = 2m+1, g(n) = 4n+5. Let C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gB, where A,B,C are nonempty.

Let n ∈ A. Then n ∈ C ∪ gB.

case 1. n ∈ C\gB. Then n ∈ B, 2n+2 ∈ fB, 2n+2 ∈ C, 2n+2 ∈ fA. This contradicts C ∩ fA = ∅.

case 2. n ∈ gB. Let n = 4m+5, m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ B. Since m < 2m+2 are from B, we have 4m+5 ∈ fB. Since 4m+5 = n ∈ A, this contradicts A ∩ fB = ∅. QED

The following pertains to 5,6'.

LEMMA 3.12.21. C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gA has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n) = 2m+1, g(n) = 4n+5. Let C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gA, where A,B,C are nonempty.

Let n = min(A). Then n ∈ C ∪ gA. Clearly n ∉ gA, n ∈ C, n ∈ B ∪ gB.

case 1. n ∈ B. Then 2n+2 ∈ fB, 2n+2 ∈ C, 2n+2 ∈ fA. This contradicts C ∩ fA = ∅.

case 2. n ∈ gB. Let n = 4m+5, m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ B. Since m < 2m+2 are from B, we have 4m+5 ∈ fB. Since 4m+5 ∈ A, this contradicts A ∩ fB = ∅. QED

The following pertains to 5,5'.

LEMMA 3.12.22. C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gC has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = 2m, f(m,n) = 4m, g(n) = 2n+1. Let C ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ C ∪. gC, where A,B,C are nonempty.

Let n ∈ A. Then 2n+2 ∈ fA, n ∈ C ∪ gC.

case 1. n ∈ C. Then n ∈ B ∪ gB.

Suppose n ∈ B. Then 2n+2 ∈ fB, 2n+2 ∈ C. Since 2n+2 ∈ fA, this contradicts C ∩ fA = ∅.

Suppose n ∈ gB. Let n = 2m+1, m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ B. Since m < 2m+2 are from B, we have 4m+4 = 2n+2

∈ fB, 2n+2 ∈ C. Since 2n+2 ∈ fA, this contradicts C ∩ fA = ∅.

case 2. n ∈ gC. Let n = 2m+1, m ∈ C, m ∈ B ∪ gB.

Suppose m ∈ B. Then 2m+2 ∈ fB, 2m+2 ∈ C, 2m+2 ∈ B. Since m < 2m+2 are from B, we have 4m+4 = 2n+2 ∈ fB, 2n+2 ∈ C. Since 2n+2 ∈ fA, this contradicts C ∩ fA = ∅.

Suppose m ∈ gB. Let m = 2r+1, r ∈ B. Then 2r+2 ∈ fB, 2r+2 ∈ C, 2r+2 ∈ B. Since r < 2r+2 are from B, we have 8r+8 = 4m+4 = 2n+2 ∈ fB, 2n+2 ∈ C. Since 2n+2 ∈ fA, this contradicts C ∩ fA = ∅.

QED

part 6. C ∪. fA ⊆ B ∪. gC.

6,1′. C ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,2′. C ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,3′. C ∪. fA ⊆ B ∪. gC, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,4′. C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,5′. C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,6′. C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.

LEMMA 3.12.23. 6,1′ - 6,3′ have ¬NON.

Proof: By Lemma 3.12.2. QED

The following pertains to 6,5′.

LEMMA 3.12.24. C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gC has ¬NON.

Proof: Let f,g ∈ ELG be defined as follows. For all n < m, let f(n,n) = 2n+2, f(n,m) = f(m,n)= 2m+1, g(n) = 4n+5. Let C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gC, where A,B,C are nonempty.

Let n ∈ A. Then n ∈ C ∪ gC, 2n+2 ∈ fA.

case 1. $n \in C$. Then $n \in B \cup gC$, $n \notin gC$, $n \in B$, $2n+2 \in fB$, $2n+2 \in C$. This contradicts $C \cap fA = \emptyset$.

case 2. $n \in gC$. Let $n = 4r+5$, $r \in C$. Then $r \in B \cup gC$, $r \in B$, $2r+2 \in fB$, $2r+2 \in C$, $2r+2 \in B \cup gC$, $2r+2 \in B$. Since $r < 2r+2$ are from B, we have $4r+5 = n \in fB$. Since $n \in A$, this contradicts $A \cap fB = \emptyset$.

QED

The following pertains to 6,4'.

LEMMA 3.12.25. $C \cup. fA \subseteq B \cup. gC$, $A \cup. fB \subseteq C \cup. gB$ has $\neg$NON.

Proof: Let $f,g \in$ ELG be defined as in the proof of Lemma 3.12.16, whose definitions we repeat here. For all $n < m$, let $f(2n,2n,2n) = f(2n+1,2n+1,2n+1) = 4n$, $f(n,m,m) = 2m$, $f(n,m,n) = 4m$, $f(m,n,n) = 8m$, $g(2n) = g(2n+1) = 4n+1$. For all other triples a,b,c, let $f(a,b,c) = 2\max(a,b,c)$. Let $C \cup. fA \subseteq B \cup. gC$, $A \cup. fB \subseteq C \cup. gB$, where A,B,C are nonempty.

Let $n = \min(A)$. Then $n \in C \cup gB$.

case 1. $n \in C$. Then $n \in B \cup gC$.

case 1a. $n \in C$, $n \in B$. Clearly $f(n,n,n) \in fB$, $f(n,n,n) \in C$. Since $f(n,n,n) \in fA$, this contradicts $C \cap fA = \emptyset$.

case 1b. $n \in C$, $n \in gC$. Let $n' = \min(C \cap gC)$. Let $n' = g(m)$, $m \in C$. Then $m \in B \cup gC$. If $m \in B$ then $n' \in gB$, which contradicts $C \cap gB = \emptyset$. Hence $m \in gC$. So $m \in C \cap gC$ and $m < n'$, which is a contradiction.

case 2. $n \in gB$. Let $n = g(m)$, $m \in B$. Then $f(m,m,m) \in fB$, $f(m,m,m) \in C$, $f(m,m,m) \in B$. So
$f(f(m,m,m),f(m,m,m),f(m,m,m)) \in fB$,
$f(f(m,m,m),f(m,m,m),f(m,m,m)) \in C$.

By the proof of Lemma 3.12.16,

$$f(f(m,m,m),f(m,m,m),f(m,m,m)) = f(g(m),g(m),g(m)) = f(n,n,n) \in fA.$$

This contradicts $C \cap fA = \emptyset$. QED

The following pertains to 6,6'.

LEMMA 3.12.26. C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gA has ¬NON.

Proof: Let f,g ∈ ELG be defined as follows. For all $n < m$, let $f(n,n,n) = 2n$, $f(n,n,m) = 2n+2$, $f(n,m,n) = 4m+2$, $f(n,m,m) = 4m-3$, $g(n) = 4n+1$. At all other triples define $f(a,b,c) = |a,b,c|+2$. Let C ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ C ∪. gA, where A,B,C are nonempty.

Let n = min(A). We claim that n ∉ B. To see this, let n ∈ B. Then 2n ∈ fB, 2n ∉ gA, 2n ∈ C, 2n ∈ fA. This contradicts C ∩ fA = ∅.

Since n ∈ C ∪ gA, we have n ∈ C, n ∈ B ∪ gC, n ∈ gC.

Let n = 4m+1, m ∈ C. Suppose m ∉ gC. Then m ∈ B, 2m ∈ fB, 2m ∈ C, 2m ∈ B. Since m,2m ∈ B, we have 4m+1 ∈ fB, 4m+1 ∈ A, contradicting A ∩ fB = ∅. Hence m ∈ gC.

Let p be greatest such that the sequence $n, g^{-1}(n), \ldots, g^{-p}(n)$ is defined and remains in C. Then $p \geq 2$.

Note that $g^{-p}(n) \in C \backslash gC$, $g^{-p}(n) \in B \cup gC$, $g^{-p}(n) \in B$. We have gone down by g^{-1}. We can go back up from $g^{-p}(n) \in B$ as follows.

First we apply the function 2n followed by the function 2n+2 (available through f(n,n,n) and f(n,n,m)). After applying the function 2n, we obtain an even element of fB, which must lie in C,B. After applying the function 2n+2, we arrive at $g^{-p+1}(n)+1$, which is also even and lies in C,B. Then we apply the function 4n+2 successively until arriving at $g^{-1}(n)+1$, which lies in C,B. Finally apply the function 4n-3, which arrives at n, and lies in fB. Since n ∈ A, we have contradicted A ∩ fB = ∅. QED