3.13. ACBC.

Recall the reduced table for AC from section 3.10.

REDUCED AC

1. A ∪. fA ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
4. B ∪. fA ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
5. B ∪. fA ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
6. B ∪. fA ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.

Recall the reduced table for BC from section 3.8.

REDUCED BC

1′. B ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
2′. B ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
3′. B ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
4′. A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
5′. A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
6′. A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

We can take advantage of symmetry through interchanging A with B as follows. Clearly (i,j′) and (j,i′) are equivalent, by interchanging A and B. So we can require that i ≤ j. Thus we have the following 21 ordered pairs to consider.

We must determine the status of all attributes INF, AL, ALF, FIN, NON, for each pair.

1,1′. A ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
1,2′. A ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.
1,3′. A ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
1,4′. A ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
1,5′. A ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.
1,6′. A ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
2,2′. A ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.

2,3′. A ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. FIN. NON.
2,4′. A ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gB. ¬INF. AL. ¬ALF. FIN. NON.
2,5′. A ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
2,6′. A ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. FIN. NON.
3,3′. A ∪. fA ⊆ C ∪. gB, B ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
3,4′. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
3,5′. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gC. ***INF***. AL. ALF. FIN. NON.
3,6′. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
4,4′. B ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gB. INF. AL. ALF. FIN. NON.
4,5′. B ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gC. ¬INF. ¬AL. ¬ALF. FIN. NON.
4,6′. B ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.
5,5′. B ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gC. INF. AL. ALF. FIN. NON.
5,6′. B ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gA. ¬INF. ¬AL. ¬ALF. FIN. NON.
6,6′. B ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gA. INF. AL. ALF. FIN. NON.

It is among the 36 ordered pairs treated here that we finally find an ordered pair that cannot be handled within RCA_0. This is pair 3,5′. In fact, here only the attribute INF requires more than RCA_0. Note that we have notated this above in large underlined bold italics. The pair 3,5′ with INF is called the Principal Exotic Case, and is treated as Proposition A in Chapters 4 and 5. The equivalence class of the Principal Exotic Case has 12 elements, and consists of the Exotic Cases.

The following pertains to 1,1′ - 6,6′.

LEMMA 3.13.1. X ∪. fY ⊆ C ∪. gZ, W ∪. fU ⊆ C ∪. gV has FIN, provided X,Y,W,U ∈ {A,B}.

Proof: Let f,g ∈ EVSD. Let A = B = {n}, where n is sufficiently large.

case 1. $f(n,...,n) = g(n,...,n)$. Let $C = \{n\}$.

case 2. $f(n,...,n) \neq g(n,...,n)$. Let $C = \{n,f(n,...,n)\}$.

In case 1, $A = B = C$, $fA = gA$, and $A \cap fA = \emptyset$. The two inclusions are identities.

In case 2, $X = Y = W = U = A = B$. So it suffices to verify that $A \cup. fA \subseteq C \cup. gZ$ and $A \cup. fA \subseteq C \cup. gV$. Note that $A \cap fA = C \cap gA = C \cap gB = C \cap gC = \emptyset$. Also $A \cup fA \subseteq C$. QED

LEMMA 3.13.2. 1,1′, 1,3′, 1,4′, 1,6′, 3,3′, 3,4′, 3,6′, 4,4′, 4,6′, 6,6′ have INF, ALF, even for EVSD.

Proof: By the AC table, $A \cup. fA \subseteq C \cup. gA$ has INF, ALF. Replace B by A in the cited ordered pairs. QED

LEMMA 3.13.3. 2,2′, 2,5′, 5,5′ have INF, ALF.

Proof: By the AC table, $A \cup. fA \subseteq C \cup. gC$ has INF, ALF. Replace B by A in the cited ordered pairs. QED

The following pertains to 1,2′, 1,5′.

LEMMA 3.13.4. $A \cup. fA \subseteq C \cup. gA$, $C \cap gC = \emptyset$ has $\neg$AL.

Proof: Define $f,g \in ELG$ as follows. For all $n < m$, let $f(n,n) = 2n$, $f(m,n) = 4m$, $f(n,m) = 4m+1$, $g(n) = 2n+1$. Let $A \cup. fA \subseteq C \cup. gA$. $C \cap gC = \emptyset$, where A,B,C have at least 2 elements. Let $n < m$ be from A.

Clearly $2m \in fA$, $4m+1 \in fA$, $2m \in C$, $2m \notin A$, $4m+1 \notin gA$, $4m+1 \in C$, $4m+1 \in gC$. This contradicts $C \cap gC = \emptyset$. QED

The following pertains to 2,3′, 2,6′.

LEMMA 3.13.5. $A \cup. fA \subseteq C \cup. gC$, $fB \subseteq C \cup. gA$ has $\neg$AL.

Proof: Define $f,g \in ELG$ as follows. For all $n < m < r$, let $f(n,n,n) = 2n$, $f(n,n,m) = 4m$, $f(n,m,n) = 4m+1$, $f(m,n,n) = 8m+1$, $g(n) = 2n+1$. Let $A \cup. fA \subseteq C \cup. gC$, $fB \subseteq C \cup. gA$, where A,B,C have at least two elements. Let $n < m$ be from B.

Note that $2m \in fB$, $2m \in C$, $4m+1 \in gC$, $4m+1 \notin C$, $4m+1 \in fB$, $4m+1 \in gA$, $2m \in A$, $4m \in fB$, $4m \in C$, $8m+1 \in gC$, $8m+1 \notin C$,

$8m+1 \in fB$, $8m+1 \in gA$, $4m \in A$, $4m \in fA$. This contradicts $A \cap fA = \emptyset$. QED

The following pertains to 2,4'.

LEMMA 3.13.6. $A \cup. fA \subseteq C \cup. gC$, $A \cup. fB \subseteq C \cup. gB$ has $\neg$INF, $\neg$ALF.

Proof: Let f be as given by Lemma 3.2.1. Let $f' \in ELG$ be given by $f'(a,b,c,d) = f(a,b,c)$ if $c = d$; $2f(a,b,c)+1$ if $c > d$; $2|a,b,c,d|+2$ if $c < d$. Let $g \in ELG$ be given by $g(n) = 2n+1$. Let $A \cup. f'A \subseteq C \cup. gC$. $A \cup. f'B \subseteq C \cup. gB$, where A,B,C have at least two elements. Let $B' = B\backslash\{\min(B)\}$. Note that $fB \subseteq f'B$.

Let $n \in fB' \cap 2N$. Then $n \in f'B \cap 2N$, $n \in C$, $2n+1 \in gC$, $2n+1 \notin C$.

We claim that $2n+1 \in f'B$. To see this, write $n = f(a,b,c)$, $a,b,c \in B'$. Then $2n+1 = f'(a,b,c,\min(B)) \in f'B$.

Hence $2n+1 \in gB$, $n \in B$, $n \in B'$. Thus we have shown that $fB' \cap 2N \subseteq B'$. Hence by Lemma 3.2.1, fB' is cofinite. Since $fB \subseteq f'B$, $f'B$ is also cofinite. Therefore B is infinite and A is finite. The former establishes $\neg$ALF, and the latter establishes $\neg$INF. QED

The following pertains to 2,4'.

LEMMA 3.13.7. $A \cup. fA \subseteq C \cup. gC$, $A \cup. fB \subseteq C \cup. gB$ has AL.

Proof: Let $f,g \in ELG$ and $p > 0$. Let $A = [n,n+p]$, where n is sufficiently large. By Lemma 3.3.3, let C be unique such that $C \subseteq [n,\infty) \subseteq C \cup. gC$. Let $B = C$.

Clearly $A \cap fA = C \cap gC = A \cap fB = C \cap gB = \emptyset$.

Since $A \cup fA \cup fB \subseteq [n,\infty)$, we have $A \cup fA \subseteq C \cup gC$, $A \cup fB \subseteq C \cup gB = C \cup gC$. Obviously $C = B$ is infinite. QED

The following pertains to 4,5'.

LEMMA 3.13.8. $B \cup. fA \subseteq C \cup. gA$, $A \cup. fB \subseteq C \cup. gC$ has $\neg$AL.

Proof: Let f be as given by Lemma 3.2.1. Let $f' \in ELG$ be defined by $f'(a,b,c,d) = f(a,b,c)$ if $c = d$; $4f(a,b,c)+3$ if c > d; 2|a,b,c,d|+2 if c < d. Let g be as given by Lemma 3.6.1. Let B ∪. f'A ⊆ C ∪. gA, A ∪. f'B ⊆ C ∪. gC, where A,B,C have at least two elements. Let A' = A\{min(A)}.

Let n ∈ fA' ∩ 2N. Then n ∈ f'A ∩ 2N, n ∈ C, 4n+3 ∈ gC, 4n+3 ∉ C, 4n+3 ∈ f'A, 4n+3 ∈ gA, n ∈ A, n ∈ A'. By Lemma 3.2.1, fA' is cofinite. Since fA ⊆ f'A, we see that f'A is cofinite.

We have established that C ∪ gA is cofinite and C ∩ gC = ∅. Hence by Lemma 3.6.1, C ⊆ A. Since fB contains an even element 2r, we have 2r ∈ C,A,f'B. This contradicts A ∩ f'B = ∅. QED

The following pertains to 5,6'.

LEMMA 3.13.9. B ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gA has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = f(m,n) = 4m+1, g(n) = 2n+1. Let B ∪. fA ⊆ C ∪. gC. A ∪. fB ⊆ C ∪. gA, where A,B,C have at least two elements. Let n < m be from B.

Clearly 2m ∈ fB, 2m ∈ C, 4m+1 ∈ gC, 4m+1 ∉ C, 4m+1 ∈ fB, 4m+1 ∈ gA, 2m ∈ A. This contradicts A ∩ fB = ∅. QED

The following pertains to 3,5'.

LEMMA 3.13.10. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gC has ALF.

Proof: Let f,g ∈ ELG and p > 0. Let A = [n,n+p], where n is sufficiently large. By Lemma 3.3.3, let S be unique such that S ⊆ [n,∞) ⊆ S ∪. gS. Let B = S ∩ [n,max(fA)]. Let C = S ∩ [n,max(fB)].

Clearly A ∩ fA = A ∩ fB = A ∩ fS = A ∩ gS = ∅. Hence A ⊆ S. Therefore A ⊆ B, A ⊆ C, B ⊆ C. Hence A,B,C are finite and have at least p elements.

Since B,C ⊆ S, we have S ∩ gS = ∅, C ∩ gC ⊆ S ∩ gS = ∅, and C ∩ gB ⊆ S ∩ gS = ∅.

We claim fA ⊆ C ∪ gB. To see this, let m ∈ fA. Then m ∈ S ∪ gS.

case 1. $m \in S$. Then $m \in B$, $m \in C$.

case 2. $m \in gS$. Write $m = g(s_1,\ldots,s_q)$, $s_1,\ldots,s_q \in S \subseteq [n,\infty)$. Then $s_1,\ldots,s_q < m \leq \max(fA)$. Hence $s_1,\ldots,s_q \in B$. So $m \in gB$.

We claim $fB \subseteq C \cup gC$. To see this, let $m \in fB$. Then $m \in S \cup gS$.

case 3. $m \in S$. Then $m \in C$.

case 4. $m \in gS$. Write $m = g(t_1,\ldots,t_q)$, $t_1,\ldots,t_q \in S \subseteq [n,\infty)$. Then $t_1,\ldots,t_q < m \leq \max(fB)$. Hence $t_1,\ldots,t_q \in C$. So $m \in gC$. QED

The Proposition asserting that 3,5′ has INF is the subject of the next two Chapters of this book. This is the Principal Exotic Case. It is not provable in ZFC (assuming ZFC is consistent). See Definitions 3.1.1 and 3.1.2.