CHAPTER 3
6561 CASES OF EQUATIONAL BOOLEAN RELATION THEORY

In this Chapter, we study 6561 = 3^8 assertions of EBRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF) of a particularly simple form. We cannot come close to analyzing all assertions of EBRT in A,B,C,fA,fB,fC,gA,gB,gC on (ELG,INF), or even of EBRT in A,B,C,fA,fB,gB,gC,⊆ on (ELG,INF).

Recall the notation A ∪. B, introduced in Definition 1.3.1. Thus

$$A \cup. B \subseteq C \cup. D$$

means

$$A \cap B = \emptyset \wedge C \cap D = \emptyset \wedge A \cup B \subseteq C \cup D.$$

This is a very natural concept, and is illustrated by the following diagram.

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 ______________________________
| C                             |
|                               |
|        _______________        |
|       | A      | B      |        |
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|       |       |      |       |
|_______|_______|______|_______|
| D     |       |      |       |
|       |       |      |       |
|       |_______|______|       |
|                              |
|                              |
|______________________________|
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Our 6561 = 3^8 cases take the following form.

TEMPLATE. For all f,g ∈ ELG there exist A,B,C ∈ INF such that

$$X \cup. fY \subseteq V \cup. gW$$
$$P \cup. fR \subseteq S \cup. gT.$$

Here X,Y,V,W,P,R,S,T are among the three letters A,B,C. We refer to the statements X ∪. fY ⊆ V ∪. gW, for X,Y,V,W ∈ {A,B,C}, as clauses.

In this Chapter, we determine the truth values of all of these 6561 statements. We prove a number of specific results about the Template. Here "Temp" is read "Template".

TEMP 1. Every assertion in the Template is either provable or refutable in $SMAH^+$. There exist 12 assertions in the Template, provably equivalent in RCA_0, such that the remaining 6549 assertions are each provable or refutable in RCA_0. Furthermore, these 12 are provably equivalent to the 1-consistency of SMAH over ACA' (Theorem 5.9.11).

We can be specific about the 12 exceptional cases.

DEFINITION 3.1.1. The Principal Exotic Case is Proposition A below. It is an instance of the Template.

PROPOSITION A. For all f,g ∈ ELG there exist A,B,C ∈ INF such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

In Chapter 4, we prove Proposition A in $SMAH^+$. In Chapter 5, we show that Proposition A is provably equivalent to 1-Con(SMAH) over ACA'.

DEFINITION 3.1.2. The Exotic Cases consist of the 12 variants of Proposition A where we (optionally) interchange the two clauses, and (optionally) permute the three letters A,B,C.

The Principal Exotic Case is among the Exotic Cases.

The Template is based on the BRT setting (ELG,INF). What if we use (ELG $\cap$ SD,INF), (SD,INF), (EVSD,INF)?

TEMP 2. Every one of the 6561 assertions in the Template, other than the 12 Exotic Cases, are provably equivalent to the result of replacing ELG by any of ELG $\cap$ SD, SD, EVSD. All 12 Exotic Cases are refutable in RCA_0 if ELG is replaced by SD or EVSD (Theorem 6.3.5).

TEMP 3. The Template behaves very differently for MF. For example, the Template is true (even provable in RCA_0) with A $\cup$. fA $\subseteq$ B $\cup$. gB, A $\cup$. fA $\subseteq$ B $\cup$. gB, yet false (even refutable in RCA_0) with ELG replaced by MF.

DEFINITION 3.1.3. The "template attributes" are as follows. Below, α,β are clauses in the sense of the Template.

INF(α,β). For all f,g $\in$ ELG there exist A,B,C $\in$ INF such that α,β hold.

AL(α,β). For all f,g $\in$ ELG and n $\geq$ 0, there exist A,B,C $\subseteq$ N, each with at least n elements (possibly infinite), such that α,β holds. Here AL is read "arbitrarily large".

ALF(α,β). For all f,g $\in$ ELG and n $\geq$ 0, there exist finite A,B,C $\subseteq$ N, each with at least n elements, such that α,β holds. Here ALF is read "arbitrarily large finite".

FIN(α,β). For all f,g $\in$ ELG there exist nonempty finite A,B,C $\subseteq$ N such that α,β holds. Here FIN is read "nonempty finite".

NON(α,β). For all f,g $\in$ ELG there exist nonempty A,B,C $\subseteq$ N such that α,β hold. Here NON is read "nonempty".

Note that the Template is based on INF(α,β).

We write $\neg$INF(α,β), $\neg$AL(α,β), $\neg$ALF(α,β), $\neg$FIN(α,β), $\neg$NON(α,β) for the negations of the template attributes.

We analyze the following Extended Template based on the template attributes.

EXTENDED TEMPLATE. $X(\alpha,\beta)$, where $X \in$ {INF, AL, ALF, FIN, NON}, and α,β are among the $X \cup. fY \subseteq V \cup. gW$, with $X,Y,V,W \in \{A,B,C\}$.

Every assertion in the Template is an assertion in the Extended Template, using X = INF. The number of assertions in the Extended Template is obviously 5(81)(81) = 32,805.

We determine the truth value of every one of these 32,805 assertions in this Chapter.

Now 32,805 is a rather daunting number, and we take full advantage of an obvious symmetry and some general facts in order to carry out such a large tabular classification.

The obvious symmetry is that we can permute any two clauses, and also permute the three letters A,B,C. This results in an obvious equivalence relation on the ordered pairs of clauses, where the equivalence classes have at most 12 elements. In fact, the typical equivalence class has 12 elements, and we compute that there are exactly 574 equivalence classes under this equivalence relation. This equivalence relation is called the pair equivalence relation, and is introduced formally in Definition 3.1.1.

Thus, in this Chapter, we will be making a total of 5(574) = 2870 determinations up to pair equivalence.

Here is one of our main results of this Chapter. "ETEMP" is read "Extended Template".

ETEMP. Every assertion in the Extended Template, other than the 12 Exotic Cases with INF, is provable or refutable in RCA_0.

The determination of the truth value of all assertions in the Extended Template is presented in section 3.14 as an annotated table.

The annotated table lists a representative from all 574 of the equivalence classes of the ordered pairs of clauses. To its right is the sequence of template attributes INF, ALF, ALF, FIN, NON, where none, some, or all appear with a negation sign in front. There are 5(574) = 2870 entries in the annotative table.

The 12 Exotic Cases then appear as entry 28 under ACBC, in the annotated table:

28. $A \cup. fA \subseteq C \cup. gB$, $A \cup. fB \subseteq C \cup. gC$. INF. AL. ALF. FIN. NON.

This is the Principal Exotic Case. The justification of this single entry (with INF only) uses $SMAH^+$, and is given in Chapter 4.

In section 3.15, we make some observations about the classification in the annotated table of section 3.14. The most important is "BRT Transfer", which tells us that for the purposes of this Chapter, INF and ALF are equivalent.

We shall see that BRT Transfer is itself provably equivalent to the 1-consistency of SMAH over ACA'.

3.1. Preliminaries.

We begin with two background Theorems which show the equivalence of

ELG and ELG ∩ SD.
SD and EVSD.

for the Extended Template.

THEOREM 3.1.1. Suppose that for all $f,g \in$ ELG ∩ SD there exist $A,B,C \in$ INF such that $X \cup. fY \subseteq V \cup. gW$ and $P \cup. fR \subseteq S \cup. gT$. Then for all $f,g \in$ ELG there exist $A,B,C \in$ INF such that $X \cup. fY \subseteq V \cup. gW$ and $P \cup. fR \subseteq S \cup. gT$. The same is true if we replace "$A,B,C \in$ INF" by "arbitrarily large $A,B,C \subseteq N$", "arbitrarily large finite $A,B,C \subseteq N$", "nonempty finite $A,B,C \subseteq N$", or "nonempty $A,B,C \subseteq N$".

Proof: Assume the hypothesis. Let $f,g \in$ ELG, with arities p,q, respectively. Let f,g be strictly dominating on $[n,\infty)^p$ and $[n,\infty)^q$, respectively.

Let f',g' be defined by $f'(x) = f(x+n)-n$ and $g'(y) = g(y+n)-n$. We claim that $f',g' \in$ ELG ∩ SD. To see this, first note that $f'(x) = f(x+n)-n > |x+n|-n = |x|$, and $g'(y)$

$= g(y+n)-n > |y+n|-n = |y|$. Hence $f',g' \in SD$. Now let $1 < c < d$ be such that

$$c|x| \leq f(x) \leq d|x|$$
$$c|y| \leq g(y) \leq d|y|$$

hold for all $|x|,|y| > t$. Then

$$c|x+n| \leq f(x+n) = f'(x)+n \leq d|x+n|$$
$$c|y+n| \leq g(y+n) = g'(y)+n \leq d|y+n|$$

hold for all $|x|,|y| > t$. Hence

$$c|x+n|-n \leq f'(x) \leq d|x+n|-n$$
$$c|y+n|-n \leq g'(y) \leq d|y+n|-n$$

hold for all $|x|,|y| > t$.

Hence

$$c|x| \leq f'(x) \leq d(|x|+n)-n = d|x|+(d-1)n \leq (d+dn)|x|$$
$$c|y| \leq g'(y) \leq d(|y|+n)-n = d|y|+(d-1)n \leq (d+dn)|y|$$

hold for all $|x|,|y| > t$. Hence $f',g' \in ELG$. So $f',g' \in ELG \cap SD$.

Applying the hypothesis to f',g', let $A,B,C \in INF$, where $X \cup. f'Y \subseteq V \cup. g'W$ and $P \cup. fR \subseteq S \cup. gT$. Let $A' = A+n$, $B' = B+n$, $C' = C+n$. Obviously $A',B',C' \in INF$, and $X',Y',V',W',P',R',S',T' = X+n,Y+n,V+n,W+n,P+n,R+n,S+n,T+n$, respectively, also lie in INF.

We claim that for all $E \subseteq N$, $f(E+n) = (f'E)+n$. To see this, note that $r \in f(E+n) \leftrightarrow (\exists x \in E)(r = f(x+n)) \leftrightarrow (\exists x \in E)(r-n = f(x+n)-n) \leftrightarrow (\exists x \in E)(r-n = f'(x)) \leftrightarrow r-n \in f'E \leftrightarrow r \in (f'E)+n$. Analogously, $g(E+n) = (g'E)+n$.

We now have

$X' \cup fY' = X+n \cup f(Y+n) = X+n \cup (f'Y)+n = (X \cup f'Y)+n \subseteq (V \cup g'W)+n = V+n \cup (g'W)+n = V+n \cup g(W+n) = V' \cup gW'$.

$X' \cap fY' = X+n \cap f(Y+n) = X+n \cap (f'Y)+n = (X \cap f'Y)+n = \emptyset$.
$V' \cap gW' = V+n \cap g(W+n) = V+n \cap (g'W)+n = (V \cap g'W)+n = \emptyset$.

The second clause

$$P' \cup. fR' \subseteq S' \cup. gT'$$

is verified in the same way.

For the other four attributes, note that for all E ⊆ N, E and E+n have the same cardinality. QED

Theorem 3.1.1 does not mean that ELG and ELG ∩ SD behave the same way in other BRT contexts - e.g., in EBRT in A,fA. Nor does it mean that EVSD and SD behave the same way in other BRT contexts, either.

In fact, consider the equation A ∪. fA = U (the Complementation Theorem). This equation is correct in EBRT in A,fA on (SD,INF), but incorrect in EBRT in A,fA on (ELG,INF). The function f(x) = 2x serves as a counterexample.

THEOREM 3.1.2. Suppose that for all f,g ∈ SD there exist A,B,C ∈ INF such that X ∪. fY ⊆ V ∪. gW and P ∪. fR ⊆ S ∪. gT. Then for all f,g ∈ EVSD there exist A,B,C ∈ INF such that X ∪. fY ⊆ V ∪. gW and P ∪. fR ⊆ S ∪. gT. The same is true if we replace "A,B,C ∈ INF" by "arbitrarily large A,B,C ⊆ N", "arbitrarily large finite A,B,C ⊆ N", "nonempty finite A,B,C ⊆ N", or "nonempty A,B,C ⊆ N".

Proof: Follow the proof of Theorem 3.1.1. The only difference between the proofs is that here we need only verify that if f,g ∈ EVSD then f',g' ∈ SD. QED

We have observed that ELG, ELG ∩ SD, SD, EVSD behave the same with regard to the Template (i.e., with INF), except for the Exotic Cases (see Theorem 6.3.5). Thus in this Chapter, we will be using EVSD whenever we are proving INF.

We know that ELG (or equivalently, ELG ∩ SD) and SD (or equivalently EVSD) do behave differently on some of the five attributes, even with the non Exotic Cases. See Theorem 3.3.10 for an example with the attribute AL.

Note that there are exactly 81 clauses and $81^2 = 3^8 = 6561$ ordered pairs of clauses used in the Template. This is a large number of cases to analyze, and so we will take full advantage of whatever shortcuts we can find.

The main shortcut that we use very effectively is syntactic equivalence. We also need to make sure that we in fact determine all 6561 truth values, without leaving any cases out. This requires some effective organization of the work.

DEFINITION 3.1.4. We say that (α,β) and (γ,δ) are pair equivalent if and only if there is a permutation π of $\{A,B,C\}$ such that

i) $\pi\alpha = \gamma \wedge \pi\beta = \delta$; or
ii) $\pi\alpha = \delta \wedge \pi\beta = \gamma$.

Obviously, if two ordered pairs of clauses are pair equivalent then the truth values of the corresponding Template statements are the same.

In this section, we generate a unique set of representatives for all the equivalence classes under the ordered pair equivalence relation. These representatives are organized into 11 groups that correspond to sections 3.3 - 3.13.

We find that there are a total of 574 equivalence classes under the pair equivalence relation. In sections 3.3 - 3.13, we determine the truth values of the 574 corresponding Template statement, within RCA_0, with the one exception of the Exotic Cases.

Section 3.14 annotates the set of representatives constructed in this section with these truth values. Section 3.15 is devoted to observed facts about the classification in section 3.14.

LEMMA 3.1.3. The following is provable in RCA_0. Let $(\alpha,\beta),(\gamma,\delta)$ be two pair equivalent ordered pairs of clauses, and let P be any one of our five attributes INF, AL, ALF, FIN, NON. Then $P(\alpha,\beta) \leftrightarrow P(\gamma,\delta)$. Moreover, if $\alpha = \beta$ then $P(\alpha) \leftrightarrow P(\beta) \leftrightarrow P(\alpha,\alpha) \leftrightarrow P(\beta,\beta)$.

Proof: Obvious. QED

Let the ordered pair of clauses

$$\alpha = X \cup. fY \subseteq V \cup. gW$$
$$\beta = P \cup. fR \subseteq S \cup. gT$$

be given.

DEFINITION 3.1.5. The inner (outer) trace of (α,β) is YVRS (XWPT).

We also consider traces independently of ordered pairs of clauses.

DEFINITION 3.1.6. A trace is a length 4 string from {A,B,C}. There are $3^4 = 81$ traces.

DEFINITION 3.1.7. Let XYVW be a trace. The reverse of XYVW is VWXY.

Any permutation π of {A,B,C} transforms any trace XYVW to the trace $\pi X\pi Y\pi V\pi W$.

DEFINITION 3.1.8. Two traces are equivalent if and only if there is a permutation that transforms the first into the second, or a permutation that transforms the first into the reverse of the second.

Equivalence of inner (outer) traces is easily seen to be an equivalence relation.

LEMMA 3.1.4. If two ordered pairs of clauses are pair equivalent, then their inner (outer) traces are equivalent.

Proof: Obvious. QED

LEMMA 3.1.5. Every trace is equivalent to exactly one of the following traces.
1. AAAA.
2. AAAB.
3. AABA.
4. AABB.
5. AABC.
6. ABAB.
7. ABAC.
8. ABBA.
9. ABBC.
10. ABCB.

Proof: We first show that every trace is equivalent to at least one of these 10. Let α be a trace. We go through a series of steps resulting in one of these 10.

First permute α so that the first term is A. Next, if the second term is C, interchange C with B so that the first two terms are AB. Note that the first two terms are AA or AB.

We now split into cases according to the first three terms.

case 1. AAA. Note that AAAA and AAAB are already on the list. AAAC is equivalent to AAAB.

case 2. AAB. Note that all three continuations are on the list.

case 3. AAC. Permute C and B, and apply case 2.

case 4. ABA. ABAB and ABAC are on the list. ABAA is the reversal of AAAB, and hence ABAA is equivalent to AAAB. AAAB is on the list.

case 5. ABB. ABBA and ABBC are on the list. ABBB is equivalent to BBAB and to AABA, which is on the list.

case 6. ABC. ABCA is equivalent to CAAB and to ABBC, which is on the list. ABCB is on the list. ABCC is equivalent to CCAB and to AABC, which is on the list.

Now we show that all 10 are inequivalent.

1. AAAA. This has the following property preserved under equivalence: just one letter is used. The remaining 9 do not have this property.
2. AAAB. This has the following property preserved under equivalence: there are exactly three equal letters and the first and third letters are the same. The remaining 9 do not have this property.
3. AABA. This has the following property preserved under equivalence: there are exactly three equal letters and the first and third letters are different. The remaining 9 do not have this property.
4. AABB. This has the following property preserved under equivalence: the first two letters equal, the last two letters are equal, and not all letters are equal. The remaining 9 do not have this property.
5. AABC. This has the following property preserved under equivalence: all three letters are used, and either the first two letters are the same, or the last two letters are the same. The remaining 9 do not have this property.
6. ABAB. This has the following property preserved under equivalence: the first and third letters are the same, the

second and fourth letters are the same, and not all letters are equal. The remaining 9 do not have this property.
7. ABAC. This has the following property preserved under equivalence: all three letters are used, where the first and third letters are equal. The remaining 9 do not have this property.
8. ABBA. This has the following property preserved under equivalence: the first and last letters are equal, the middle two letters are equal, and not all letters are equal. The remaining 9 do not have this property.
9. ABBC. This has the following property preserved under equivalence: all three letters are used, where the two middle letters are equal, or the first and last letters are equal. The remaining 9 do not have this property.
10. ABCB. This has the following property preserved under equivalence: all three letters are used, where the second and fourth letters are equal. The remaining 9 do not have this property.

QED

LEMMA 3.1.6. Every ordered pair of clauses is pair equivalent to an ordered pair of clauses whose inner trace is among
1. AAAA.
2. AAAB.
3. AABA.
4. AABB.
5. AABC.
6. ABAB.
7. ABAC.
8. ABBA.
9. ABBC.
10. ACBC.

Proof: Immediate from Lemma 3.1.5. Note that we have changed item 10 from Lemma 1.3 by interchanging B and C. The reason for this change is that the inner trace of the ordered pair of clauses in Proposition A is ACBC, and we like the exact choice of letters in Proposition A. QED

In section 3.3 we handle all of the duplicates (α,α). We remove these duplicates from consideration in sections 3.4 and 3.9 where they arise. Obviously, they do not arise in the remaining sections.

We now wish to give the unique set of representatives of the pair equivalence classes of the ordered pairs of clauses that we use to tabulate our results in the annotated tables of section 3.14.

To support our choice of unique representatives, we establish a number of facts.

We have been working with pair equivalence, and inner (outer) traces, for ordered pairs of clauses. It is convenient to have these notions for a single clause:

DEFINITION 3.1.9. Two individual clauses are considered equivalent if and only if there is a permutation of {A,B,C} then sends one to the other. The inner (outer) trace of the single clause X ∪. fY ⊆ V ∪. gW. is defined to be YV, XW, respectively.

LEMMA 3.1.7. Every clause is equivalent to a clause where
i) the inner trace is AA; or
ii) the inner trace is AB.
No clause in one of these two categories is equivalent to any clause in the other of these two categories.

Proof: If the inner trace begins with B or C, then permute it with A, so that the inner trace now begins with A. If the inner trace is AC, then permute C and B, so that the inner trace is now AB.

Let π be a permutation of {A,B,C}. It is clear that π must map any clause with inner trace AA to a clause with trace XX. Hence no clause in category i) can be equivalent to a clause in category ii). Also, π must map any clause with inner trace AB to a clause with inner trace XY, X $\neq$ Y. Hence no clause in category ii) can be equivalent to a clause in category i). QED

Lemma 3.1.7 supports the unique set of representatives of individual clauses (or, equivalently, duplicates), presented in the following way.

We list all individual clauses according to Lemma 3.1.7, ordered first by the two categories i),ii), and then lexicographically (reading the four letters from left to right). These are consecutively numbered starting with 1. But if a clause is equivalent to some earlier clause, then we label it with an x, and also point to the earlier numbered clause to which it is equivalent.

SINGLE CLAUSES (14)

1. $A \cup. fA \subseteq A \cup. gA$.
2. $A \cup. fA \subseteq A \cup. gB$.
x. $A \cup. fA \subseteq A \cup. gC$. ≡ 2.

3. $B \cup. fA \subseteq A \cup. gA$.
4. $B \cup. fA \subseteq A \cup. gB$.
5. $B \cup. fA \subseteq A \cup. gC$.

x. $C \cup. fA \subseteq A \cup. gA$. ≡ 3.
x. $C \cup. fA \subseteq A \cup. gB$. ≡ 5.
x. $C \cup. fA \subseteq A \cup. gC$. ≡ 4.

6. $A \cup. fA \subseteq B \cup. gA$.
7. $A \cup. fA \subseteq B \cup. gB$.
8. $A \cup. fA \subseteq B \cup. gC$.

9. $B \cup. fA \subseteq B \cup. gA$.
10. $B \cup. fA \subseteq B \cup. gB$.
11. $B \cup. fA \subseteq B \cup. gC$.

12. $C \cup. fA \subseteq B \cup. gA$.
13. $C \cup. fA \subseteq B \cup. gB$.
14. $C \cup. fA \subseteq B \cup. gC$.

The numbered part of this table, annotated, appears in section 3.14.

DEFINITION 3.1.10. An AAAA ordered pair is an ordered pair of **distinct** clauses whose inner trace is AAAA. We also use this terminology for the other 9 traces in Lemma 3.1.6 (which are the titles of sections 3.5 - 3.13).

Thus we are using the ten inner traces of Lemma 3.1.6 to divide the (non duplicate) ordered pairs mod pair equivalence into ten more manageable categories. The ordered pairs within each category have different outer traces.

LEMMA 3.1.8. Every AAAA ordered pair is pair equivalent to an AAAA ordered pair whose outer trace
i) uses exactly A,B; or
ii) uses exactly B,C, with outer trace beginning with B; or

iii) uses exactly A,B,C, whose outer trace begins with AA,AB, or B.
No ordered pair of clauses in any one of these three categories is pair equivalent to any ordered pair of clauses in any other of these categories.

Proof: Let α be an AAAA ordered pair. The outer trace of α cannot use exactly one letter, since then the items in the ordered pair would be identical.

Suppose the outer trace of α uses exactly B or exactly C. Then the two components of β are the same, which is impossible.

Suppose the outer trace of α uses exactly A,C. By interchanging B,C, we obtain an AAAA ordered pair whose outer trace uses exactly A,B.

Suppose the outer trace of α uses exactly B,C, with outer trace beginning with C. By interchanging B,C, we obtain an AAAA ordered pair whose outer trace uses exactly B,C, and which begins with B.

Suppose the outer trace of α uses exactly A,B,C, and begins with C. By interchanging B,C, we obtain an AAAA ordered pair whose outer trace uses exactly A,B,C, and which begins with B.

Suppose the outer trace of α uses exactly A,B,C, and begins with AC. By interchanging B,C, we obtain an AAAA ordered pair whose outer trace uses exactly A,B,C, and which begins with AB.

Note that categories i)-iii) list all possibilities other than the ones in the previous five paragraphs. Hence i)-iii) is inclusive.

Now suppose $\alpha \neq \beta$ be pair equivalent AAAA ordered pairs. Let π transform α to β. Then $\pi A = A$. Clearly π cannot take us from an ordered pair in any category i)-iii) to any ordered pair in a different category i)-iii). This establishes the final claim. QED

Lemma 3.1.8 supports the unique set of representatives for AAAA ordered pairs, presented in the following way.

We list all ordered pairs of clauses according to Lemma 3.1.8, ordered first by the four categories i)-iii), and then lexicographically (reading the eight letters from left to right). These are consecutively numbered starting with 1. But if an ordered pair is pair equivalent to some earlier ordered pair, then we label it with an x, and also point to the earlier numbered ordered pair of clauses to which it is pair equivalent.

In fact, the previous paragraph describes exactly how we will present the ordered pairs according to later Lemmas.

AAAA (20)

1. A ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ A ∪. gB.
2. A ∪. fA ⊆ A ∪. gA, B ∪. fA ⊆ A ∪. gA.
3. A ∪. fA ⊆ A ∪. gA, B ∪. fA ⊆ A ∪. gB.

x. A ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ A ∪. gA. ≡ 1.
4. A ∪. fA ⊆ A ∪. gB, B ∪. fA ⊆ A ∪. gA.
5. A ∪. fA ⊆ A ∪. gB, B ∪. fA ⊆ A ∪. gB.

x. B ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ A ∪. gA. ≡ 2.
x. B ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ A ∪. gB. ≡ 4.
6. B ∪. fA ⊆ A ∪. gA, B ∪. fA ⊆ A ∪. gB.

x. B ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ A ∪. gA. ≡ 3.
x. B ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ A ∪. gB. ≡ 5.
x. B ∪. fA ⊆ A ∪. gB, B ∪. fA ⊆ A ∪. gA. ≡ 6.

7. B ∪. fA ⊆ A ∪. gB, B ∪. fA ⊆ A ∪. gC.
8. B ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ A ∪. gB.
9. B ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ A ∪. gC.

x. B ∪. fA ⊆ A ∪. gC, B ∪. fA ⊆ A ∪. gB. ≡ 7.
10. B ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ A ∪. gB.
x. B ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ A ∪. gC. ≡ 8.

11. A ∪. fA ⊆ A ∪. gA, B ∪. fA ⊆ A ∪. gC.
x. A ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ A ∪. gB. ≡ 11.

12. A ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ A ∪. gC.
13. A ∪. fA ⊆ A ∪. gB, B ∪. fA ⊆ A ∪. gC.
14. A ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ A ∪. gA.
15. A ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ A ∪. gB.
16. A ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ A ∪. gC.

x. B ∪. fA ⊆ A ∪. gA, A ∪. fA ⊆ A ∪. gC. ≡ 14.
17. B ∪. fA ⊆ A ∪. gA, B ∪. fA ⊆ A ∪. gC.
18. B ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ A ∪. gA.
19. B ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ A ∪. gB.
20. B ∪. fA ⊆ A ∪. gA, C ∪. fA ⊆ A ∪. gC.

x. B ∪. fA ⊆ A ∪. gB, A ∪. fA ⊆ A ∪. gC. ≡ 16.
x. B ∪. fA ⊆ A ∪. gB, C ∪. fA ⊆ A ∪. gA. ≡ 20.

x. B ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ A ∪. gA. ≡ 11.
x. B ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ A ∪. gB. ≡ 13.
x. B ∪. fA ⊆ A ∪. gC, A ∪. fA ⊆ A ∪. gC. ≡ 15.
x. B ∪. fA ⊆ A ∪. gC, B ∪. fA ⊆ A ∪. gA. ≡ 17.
x. B ∪. fA ⊆ A ∪. gC, C ∪. fA ⊆ A ∪. gA. ≡ 19.

The numbered part of this AAAA table, annotated, appears in section 3.14.

LEMAM 3.1.9. No AAAB ordered pair is pair equivalent to any other AAAB ordered pair. All 81 AAAB ordered pairs are pair inequivalent.

Proof: Let $\alpha \neq \beta$ be AAAB ordered pairs. First suppose π transforms α to β. Then $\pi A = A$ and $\pi B = B$. Hence π is the identity, and $\alpha = \beta$.

Now suppose π transforms α to the reverse of β. Note that the reverse of β is an ABAA ordered pair. Then $\pi A = A$ and $\pi A = B$, which is impossible. QED

Since all 81 AABA ordered pairs are pair inequivalent, there is no point in listing them here. The annotated AAAB table appears in section 3.14.

LEMMA 3.1.10. No AABA ordered pair is pair equivalent to any other AABA ordered pair. All 81 AABA ordered pairs are pair inequivalent.

Proof: Let $\alpha \neq \beta$ be AABA ordered pairs. First suppose π transforms α to β. Then $\pi A = A$ and $\pi B = B$. Hence π is the identity, and $\alpha = \beta$.

Now suppose π transforms α to the reverse of β. Note that the reverse of β is a BAAA ordered pair. Then $\pi A = B$ and $\pi A = A$, which is impossible. QED

The AABA table, annotated, appears in section 3.14.

LEMMA 3.1.11. Every AABB ordered pair is pair equivalent to an AABB ordered pair whose outer trace
i) uses exactly A; or
ii) uses exactly C; or
iii) uses exactly A,B; or
iv) uses exactly A,C; or
v) uses exactly A,B,C.
No ordered pair in any one of these 5 categories is pair equivalent to a ordered pair in any other category.

Proof: Let α be an AABB ordered pair. Suppose the outer trace of α uses exactly B. By interchanging A,B, we obtain a BBAA ordered pair β whose outer trace uses exactly A. Note that the reverse of β is an AABB ordered pair whose outer trace uses exactly A.

Suppose the outer trace of α uses exactly B,C. By interchanging A,B, we obtain a BBAA ordered pair β whose outer trace uses exactly A,C. Note that the reverse of β is an AABB ordered pair whose outer trace uses exactly A,C.

Note that categories i)-v) list all possibilities other than exactly B, exactly B,C, and so i)-v) is inclusive.

Now suppose $\alpha \neq \beta$ be pair equivalent AABB ordered pairs. Let π transform α to β. Then $\pi A = A$, $\pi B = B$, and so π is the identity Hence $\alpha = \beta$, which is impossible. Let π transform α to the reverse of β. Then π interchanges A,B. Clearly π cannot take us from an ordered pair in any category i)-v) to any ordered pair in a different category i)-v). This establishes the final claim. QED

We now list the AABB ordered pairs in our by now standard way.

AABB (45)

1. $A \cup. fA \subseteq A \cup. gA$, $A \cup. fB \subseteq B \cup. gA$.
2. $C \cup. fA \subseteq A \cup. gC$, $C \cup. fB \subseteq B \cup. gC$.

3. $A \cup. fA \subseteq A \cup. gA$, $A \cup. fB \subseteq B \cup. gB$.
4. $A \cup. fA \subseteq A \cup. gA$, $B \cup. fB \subseteq B \cup. gA$.
5. $A \cup. fA \subseteq A \cup. gA$, $B \cup. fB \subseteq B \cup. gB$.

6. $A \cup. fA \subseteq A \cup. gB$, $A \cup. fB \subseteq B \cup. gA$.
7. $A \cup. fA \subseteq A \cup. gB$, $A \cup. fB \subseteq B \cup. gB$.

8. A ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ B ∪. gA.
x. A ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ B ∪. gB. ≡ 4.

9. B ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gA.
10. B ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gB.
x. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gA. ≡ 7.
x. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gB. ≡ 3.

11. B ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gA.
x. B ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gB. ≡ 9.
x. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ B ∪. gA. ≡ 6.

12. A ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gC.
13. A ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gA.
14. A ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gC.

15. A ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gA.
16. A ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gC.
17. A ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ B ∪. gA.
18. A ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ B ∪. gC.

19. C ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gA.
20. C ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gC.
21. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gA.
22. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gC.

23. C ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gA.
24. C ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gC.
25. C ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ B ∪. gA.

26. A ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gC.
27. A ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gB.

28. A ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gC.
29. A ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ B ∪. gC.
30. A ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ B ∪. gA.
31. A ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ B ∪. gB.
32. A ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ B ∪. gC.

33. A ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gB.
x. A ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ B ∪. gA. ≡ 29.
x. A ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ B ∪. gB. ≡ 26.
34. A ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ B ∪. gC.
35. A ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ B ∪. gB.

36. B ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gC.
x. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gC. ≡ 33.

37. B ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gA.
38. B ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gB.
39. B ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gC.

40. B ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gC.
41. B ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ B ∪. gA.

x. B ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gA. ≡ 40.
x. B ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gB. ≡ 36.
42. B ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gC.
x. B ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ B ∪. gA. ≡ 28.
43. B ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ B ∪. gA.

x. C ∪. fA ⊆ A ∪. gA, A ∪. fB ⊆ B ∪. gB. ≡ 38.
x. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gA. ≡ 31.
x. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gB. ≡ 27.
x. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ B ∪. gC. ≡ 35.
44. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ B ∪. gB.

x. C ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gA. ≡ 41.
x. C ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gB. ≡ 37.
x. C ∪. fA ⊆ A ∪. gB, A ∪. fB ⊆ B ∪. gC. ≡ 43.
x. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ B ∪. gA. ≡ 30.
45. C ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ B ∪. gA.

x. C ∪. fA ⊆ A ∪. gC, A ∪. fB ⊆ B ∪. gB. ≡ 39.
x. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ B ∪. gA. ≡ 32.

The numbered part of this AABB table, annotated, appears in section 3.14.

LEMMA 3.1.12. No AABC ordered pair is pair equivalent to any other AABC ordered pair. All 81 AABC ordered pairs are pair inequivalent.

Proof: Let $\alpha \neq \beta$ be AABC ordered pairs. First suppose π transforms α to β. Then $\pi A = A$, $\pi B = B$, $\pi C = C$. Hence π is the identity, and $\alpha = \beta$.

Now suppose π transforms α to the reverse of β. Note that the reverse of β is a BCAA ordered pair. Then $\pi A = B$ and $\pi A = C$, which is impossible. QED

The AABC table, annotated, appears in section 3.14.

LEMMA 3.1.13. Two distinct ABAB ordered pairs α,β are pair equivalent if and only if the reverse of α is β.

Proof: Let $\alpha \neq \beta$ be pair equivalent ABAB ordered pairs. Let π transform α to β. Then $\pi A = A$, $\pi B = B$, and so π is the identity. Hence $\alpha = \beta$,

Now suppose π transforms α to the reverse of β. Then again π is the identity, and so α is the reverse of β. QED

ABAB (36)

1. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gB.
2. A ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gC.
3. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gA.
4. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gB.
5. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gC.
6. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gA.
7. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB.
8. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gC.

x. A ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gA. ≡ 1.
9. A ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gC.
10. A ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gA.
11. A ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gB.
12. A ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gC.
13. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gA.
14. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gB.
15. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gC.

x. A ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gA. ≡ 2.
x. A ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gB. ≡ 9.
16. A ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gA.
17. A ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gB.
18. A ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gC.
19. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gA.
20. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gB.
21. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gC.

x. B ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gA. ≡ 3.
x. B ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gB. ≡ 10.
x. B ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gC. ≡ 16.
22. B ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gB.
23. B ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gC.
24. B ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gA.
25. B ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB.
26. B ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gC.

x. B ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gA. ≡ 4.

x. B ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gB. ≡ 11.
x. B ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gC. ≡ 17.
x. B ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gA. ≡ 22.
27. B ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gC.
28. B ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gA.
29. B ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gB.
30. B ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gC.

x. B ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gA. ≡ 5.
x. B ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gB. ≡ 12.
x. B ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gC. ≡ 18.
x. B ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gA. ≡ 23.
x. B ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gB. ≡ 27.
31. B ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gA.
32. B ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gB.
33. B ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gC.

x. C ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gA. ≡ 6.
x. C ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gB. ≡ 13.
x. C ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ B ∪. gC. ≡ 19.
x. C ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gA. ≡ 24.
x. C ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gB. ≡ 28.
x. C ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ B ∪. gC. ≡ 31.
34. C ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gB.
35. C ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ B ∪. gC.

x. C ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gA. ≡ 7.
x. C ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gB. ≡ 14.
x. C ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ B ∪. gC. ≡ 20.
x. C ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gA. ≡ 25.
x. C ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gB. ≡ 29.
x. C ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ B ∪. gC. ≡ 32.
x. C ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gA. ≡ 34.
36. C ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ B ∪. gC.

x. C ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gA. ≡ 8.
x. C ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gB. ≡ 15.
x. C ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ B ∪. gC. ≡ 21.
x. C ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gA. ≡ 26.
x. C ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gB. ≡ 30.
x. C ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ B ∪. gC. ≡ 33.
x. C ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gA. ≡ 35.
x. C ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ B ∪. gB. ≡ 36.

The numbered part of this ABAB table, annotated, appears in section 3.14.

LEMMA 3.1.14. Every ABAC ordered pair is pair equivalent to an ABAC ordered pair whose outer trace
i) uses exactly A; or
ii) uses exactly B; or
iii) uses exactly A,B; or
iv) uses exactly B,C; or
v) uses exactly A,B,C.
No ordered pair in any one of these 5 categories is pair equivalent to an ordered pair in any other category.

Proof: Let α be an ABAC ordered pair. Suppose the outer trace of α uses exactly C. By interchanging B,C, we obtain a pair equivalent ACAB ordered pair whose outer trace uses exactly B. Its reverse is a pair equivalent ABAC ordered pair whose outer trace uses exactly B.

Suppose the outer trace of α uses exactly A,C. By interchanging B,C, we obtain a pair equivalent ACAB ordered pair whose outer trace uses exactly A,B. Its reverse is a pair equivalent ABAC ordered pair whose outer trace uses exactly A,B.

Note that categories i)-v) list all possibilities other than exactly C, exactly A,C, and so i)-v) is inclusive.

Now suppose $\alpha \neq \beta$ are pair equivalent ABAC ordered pairs. Let π transform α to β. Then $\pi A = A$, $\pi B = B$, $\pi C = C$, and so π is the identity. Hence $\alpha = \beta$, which is impossible. Let π transform α to the reverse of β. Then π interchanges B,C. Clearly π cannot take us from an ordered pair in any category i)-v) to any ordered pair in a different category i)-v). This establishes the final claim. QED

ABAC (45)

1. $A \cup. fA \subseteq B \cup. gA$, $A \cup. fA \subseteq C \cup. gA$.
2. $B \cup. fA \subseteq B \cup. gB$, $B \cup. fA \subseteq C \cup. gB$.

3. $A \cup. fA \subseteq B \cup. gA$, $A \cup. fA \subseteq C \cup. gB$.
4. $A \cup. fA \subseteq B \cup. gA$, $B \cup. fA \subseteq C \cup. gA$.
5. $A \cup. fA \subseteq B \cup. gA$, $B \cup. fA \subseteq C \cup. gB$.

6. $A \cup. fA \subseteq B \cup. gB$, $A \cup. fA \subseteq C \cup. gA$.
7. $A \cup. fA \subseteq B \cup. gB$, $A \cup. fA \subseteq C \cup. gB$.
8. $A \cup. fA \subseteq B \cup. gB$, $B \cup. fA \subseteq C \cup. gA$.
9. $A \cup. fA \subseteq B \cup. gB$, $B \cup. fA \subseteq C \cup. gB$.

10\. B ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gA.
11\. B ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gB.
12\. B ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gA.
13\. B ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gB.

14\. B ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ C ∪. gA.
15\. B ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ C ∪. gB.
16\. B ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ C ∪. gA.

17\. B ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ C ∪. gC.
18\. B ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gB.
19\. B ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gC.

20\. B ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gB.
21\. B ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gC.
22\. B ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ C ∪. gB.
x. B ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ C ∪. gC. ≡ 18.

23\. C ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ C ∪. gB.
24\. C ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ C ∪. gC.
x. C ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gB. ≡ 21.
x. C ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gC. ≡ 17.

25\. C ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gB.
x. C ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gC. ≡ 23.
x. C ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ C ∪. gB. ≡ 20.

26\. A ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gC.
27\. A ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ C ∪. gB.

28\. A ∪. fA ⊆ B ∪. gB, A ∪. fA ⊆ C ∪. gC.
29\. A ∪. fA ⊆ B ∪. gB, B ∪. fA ⊆ C ∪. gC.
30\. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gA.
31\. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gB.
32\. A ∪. fA ⊆ B ∪. gB, C ∪. fA ⊆ C ∪. gC.

33\. A ∪. fA ⊆ B ∪. gC, A ∪. fA ⊆ C ∪. gB.
34\. A ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gA.
35\. A ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gB.
36\. A ∪. fA ⊆ B ∪. gC, B ∪. fA ⊆ C ∪. gC.
37\. A ∪. fA ⊆ B ∪. gC, C ∪. fA ⊆ C ∪. gB.

x. B ∪. fA ⊆ B ∪. gA, A ∪. fA ⊆ C ∪. gC. ≡ 30.
38\. B ∪. fA ⊆ B ∪. gA, B ∪. fA ⊆ C ∪. gC.
39\. B ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ C ∪. gA.
40\. B ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ C ∪. gB.
41\. B ∪. fA ⊆ B ∪. gA, C ∪. fA ⊆ C ∪. gC.

x. B $\cup$. fA $\subseteq$ B $\cup$. gB, A $\cup$. fA $\subseteq$ C $\cup$. gC. ≡ 32.
x. B $\cup$. fA $\subseteq$ B $\cup$. gB, C $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 41.

x. B $\cup$. fA $\subseteq$ B $\cup$. gC, A $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 27.
x. B $\cup$. fA $\subseteq$ B $\cup$. gC, A $\cup$. fA $\subseteq$ C $\cup$. gB. ≡ 37.
x. B $\cup$. fA $\subseteq$ B $\cup$. gC, A $\cup$. fA $\subseteq$ C $\cup$. gC. ≡ 31.
42. B $\cup$. fA $\subseteq$ B $\cup$. gC, B $\cup$. fA $\subseteq$ C $\cup$. gA.
x. B $\cup$. fA $\subseteq$ B $\cup$. gC, C $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 40.

x. C $\cup$. fA $\subseteq$ B $\cup$. gA, A $\cup$. fA $\subseteq$ C $\cup$. gB. ≡ 34.
43. C $\cup$. fA $\subseteq$ B $\cup$. gA, B $\cup$. fA $\subseteq$ C $\cup$. gA.
44. C $\cup$. fA $\subseteq$ B $\cup$. gA, B $\cup$. fA $\subseteq$ C $\cup$. gB.
45. C $\cup$. fA $\subseteq$ B $\cup$. gA, B $\cup$. fA $\subseteq$ C $\cup$. gC.
x. C $\cup$. fA $\subseteq$ B $\cup$. gA, C $\cup$. fA $\subseteq$ C $\cup$. gB. ≡ 42.

x. C $\cup$. fA $\subseteq$ B $\cup$. gB, A $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 26.
x. C $\cup$. fA $\subseteq$ B $\cup$. gB, A $\cup$. fA $\subseteq$ C $\cup$. gB. ≡ 36.
x. C $\cup$. fA $\subseteq$ B $\cup$. gB, A $\cup$. fA $\subseteq$ C $\cup$. gC. ≡ 29.
x. C $\cup$. fA $\subseteq$ B $\cup$. gB, B $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 45.
x. C $\cup$. fA $\subseteq$ B $\cup$. gB, C $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 38.

x. C $\cup$. fA $\subseteq$ B $\cup$. gC, A $\cup$. fA $\subseteq$ C $\cup$. gB. ≡ 35.
x. C $\cup$. fA $\subseteq$ B $\cup$. gC, B $\cup$. fA $\subseteq$ C $\cup$. gA. ≡ 44.

The numbered part of this ABAC table, annotated, appears in section 3.14.

LEMMA 3.1.15. Every ABBA ordered pair is pair equivalent to an ABBA ordered pair whose outer trace
i) uses exactly A; or
ii) uses exactly C; or
iii) uses exactly A,B; or
iv) uses exactly A,C; or
v) uses exactly A,B,C.
No ordered pair in any one of these 5 categories is pair equivalent to an ordered pair in any other category. Two distinct ABBA ordered pairs are pair equivalent if and only if the result of interchanging A,B in α is the reverse of β.

Proof: Let α be an ABBA ordered pair. Suppose the outer trace of α uses exactly B. By interchanging A,B, we obtain a pair equivalent BAAB ordered pair whose outer trace uses exactly A. Its reverse is a pair equivalent ABBA ordered pair whose outer trace uses exactly A.

Suppose the outer trace of α uses exactly B,C. By interchanging A,B, we obtain a pair equivalent BAAB ordered pair whose outer trace uses exactly A,C. Its reverse is a pair equivalent ABBA ordered pair whose outer trace uses exactly A,C.

Note that categories i)-v) list all possibilities other than exactly B, exactly B,C, and so i)-v) is inclusive.

Let $\alpha \neq \beta$ be pair equivalent ABBA ordered pairs. Let π be a permutation of {A,B,C} that transforms α to β. Then πA = A, πB = B, and so π is the identity. Hence $\alpha = \beta$, which is impossible. Suppose π transforms α to the reverse of β. Then β is a BAAB ordered pair, and so πA = B, πB = A. Clearly π cannot take us from an ordered pair in any category i)-v) to any ordered pair in a different category i)-v). This establishes the final claim. QED

ABBA (45)

1. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ A ∪. gA.
2. C ∪. fA ⊆ B ∪. gC, C ∪. fB ⊆ A ∪. gC.

3. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ A ∪. gB.
4. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gA.
5. A ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gB.

6. A ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ A ∪. gA.
7. A ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ A ∪. gB.
8. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ A ∪. gA.
x. A ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ A ∪. gB. ≡ 4.

9. B ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ A ∪. gA.
10. B ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ A ∪. gB.
x. B ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gA. ≡ 7.
x. B ∪. fA ⊆ B ∪. gA, B ∪. fB ⊆ A ∪. gB. ≡ 3.

11. B ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ A ∪. gA.
x. B ∪. fA ⊆ B ∪. gB, A ∪. fB ⊆ A ∪. gB. ≡ 9.
x. B ∪. fA ⊆ B ∪. gB, B ∪. fB ⊆ A ∪. gA. ≡ 6.

12. A ∪. fA ⊆ B ∪. gA, A ∪. fB ⊆ A ∪. gC.
13. A ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gA.
14. A ∪. fA ⊆ B ∪. gA, C ∪. fB ⊆ A ∪. gC.

15. A ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ A ∪. gA.
16. A ∪. fA ⊆ B ∪. gC, A ∪. fB ⊆ A ∪. gC.

17. $A \cup. fA \subseteq B \cup. gC, C \cup. fB \subseteq A \cup. gA.$
18. $A \cup. fA \subseteq B \cup. gC, C \cup. fB \subseteq A \cup. gC.$

19. $C \cup. fA \subseteq B \cup. gA, A \cup. fB \subseteq A \cup. gA.$
20. $C \cup. fA \subseteq B \cup. gA, A \cup. fB \subseteq A \cup. gC.$
21. $C \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gA.$
22. $C \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gC.$

23. $C \cup. fA \subseteq B \cup. gC, A \cup. fB \subseteq A \cup. gA.$
24. $C \cup. fA \subseteq B \cup. gC, A \cup. fB \subseteq A \cup. gC.$
25. $C \cup. fA \subseteq B \cup. gC, C \cup. fB \subseteq A \cup. gA.$

26. $A \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq A \cup. gC.$
27. $A \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gB.$

28. $A \cup. fA \subseteq B \cup. gB, A \cup. fB \subseteq A \cup. gC.$
29. $A \cup. fA \subseteq B \cup. gB, B \cup. fB \subseteq A \cup. gC.$
30. $A \cup. fA \subseteq B \cup. gB, C \cup. fB \subseteq A \cup. gA.$
31. $A \cup. fA \subseteq B \cup. gB, C \cup. fB \subseteq A \cup. gB.$
32. $A \cup. fA \subseteq B \cup. gB, C \cup. fB \subseteq A \cup. gC.$

33. $A \cup. fA \subseteq B \cup. gC, A \cup. fB \subseteq A \cup. gB.$
x. $A \cup. fA \subseteq B \cup. gC, B \cup. fB \subseteq A \cup. gA. \equiv 29.$
x. $A \cup. fA \subseteq B \cup. gC, B \cup. fB \subseteq A \cup. gB. \equiv 26.$
34. $A \cup. fA \subseteq B \cup. gC, B \cup. fB \subseteq A \cup. gC.$
35. $A \cup. fA \subseteq B \cup. gC, C \cup. fB \subseteq A \cup. gB.$

36. $B \cup. fA \subseteq B \cup. gA, A \cup. fB \subseteq A \cup. gC.$
x. $B \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq A \cup. gC. \equiv 33.$
37. $B \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gA.$
38. $B \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gB.$
39. $B \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gC.$

40. $B \cup. fA \subseteq B \cup. gB, A \cup. fB \subseteq A \cup. gC.$
41. $B \cup. fA \subseteq B \cup. gB, C \cup. fB \subseteq A \cup. gA.$

x. $B \cup. fA \subseteq B \cup. gC, A \cup. fB \subseteq A \cup. gA. \equiv 40.$
x. $B \cup. fA \subseteq B \cup. gC, A \cup. fB \subseteq A \cup. gB. \equiv 36.$
42. $B \cup. fA \subseteq B \cup. gC, A \cup. fB \subseteq A \cup. gC.$
x. $B \cup. fA \subseteq B \cup. gC, B \cup. fB \subseteq A \cup. gA. \equiv 28.$
43. $B \cup. fA \subseteq B \cup. gC, C \cup. fB \subseteq A \cup. gA.$

x. $C \cup. fA \subseteq B \cup. gA, A \cup. fB \subseteq A \cup. gB. \equiv 38.$
x. $C \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq A \cup. gA. \equiv 31.$
x. $C \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq A \cup. gB. \equiv 27.$
x. $C \cup. fA \subseteq B \cup. gA, B \cup. fB \subseteq A \cup. gC. \equiv 35.$
44. $C \cup. fA \subseteq B \cup. gA, C \cup. fB \subseteq A \cup. gB.$

x. $C \cup . fA \subseteq B \cup . gB, A \cup . fB \subseteq A \cup . gA.$ ≡ 41.
x. $C \cup . fA \subseteq B \cup . gB, A \cup . fB \subseteq A \cup . gB.$ ≡ 37.
x. $C \cup . fA \subseteq B \cup . gB, A \cup . fB \subseteq A \cup . gC.$ ≡ 43.
x. $C \cup . fA \subseteq B \cup . gB, B \cup . fB \subseteq A \cup . gA.$ ≡ 30.
45. $C \cup . fA \subseteq B \cup . gB, C \cup . fB \subseteq A \cup . gA.$

x. $C \cup . fA \subseteq B \cup . gC, A \cup . fB \subseteq A \cup . gB.$ ≡ 39.
x. $C \cup . fA \subseteq B \cup . gC, B \cup . fB \subseteq A \cup . gA.$ ≡ 32.

The numbered part of this ABBA list, annotated, appears in section 3.14.

LEMMA 3.1.16. No ABBC ordered pair is pair equivalent to any other ABBC ordered pair. All 81 ABBC ordered pairs are pair inequivalent.

Proof: Let $\alpha \neq \beta$ be ABBC ordered pairs. First suppose π transforms α to β. Then $\pi A = A$, $\pi B = B$, $\pi C = C$. Hence π is the identity, and $\alpha = \beta$.

Now suppose π transforms α to the reverse of β. Note that the reverse of β is a BCAB ordered pair. Then $\pi B = C$, $\pi B = A$, which is a contradiction. QED

The ABBC table, annotated, appears in section 3.14.

LEMMA 3.1.17. Every ACBC ordered pair is pair equivalent to an ACBC ordered pair whose outer trace
i) uses exactly A; or
ii) uses exactly C; or
iii) uses exactly A,C; or
iv) uses exactly A,B; or
v) uses exactly A,B,C.
No ordered pair in any one of these 5 categories is pair equivalent to an ordered pair in any other category.

Proof: Let α be an ACBC ordered pair. Suppose the outer trace of α uses exactly B. By interchanging A,B, we obtain a pair equivalent BCAC ordered pair whose outer trace uses exactly A. Its reverse is a pair equivalent ACBC ordered pair whose outer trace uses exactly A.

Suppose the outer trace of α uses exactly B,C. By interchanging A,B, we obtain a pair equivalent BCAC ordered pair whose outer trace uses exactly A,C. Its reverse is a pair equivalent ACBC ordered pair whose outer trace uses exactly A,C.

Note that categories i)-v) list all possibilities other than exactly B, exactly B,C, and so i)-v) is inclusive.

Let $\alpha \neq \beta$ be pair equivalent ACBC ordered pairs. Let π be a permutation of {A,B,C} that transforms α to β. Then πA = A, πC = C, and so π is the identity. Hence $\alpha = \beta$, which is impossible. Suppose π transforms α to the reverse of β. Then β is a BCAC ordered pair, and so πA = B, πB = A. Clearly β cannot transform any ordered pair in any category 1)-v) to any ordered pair in any different category i)-v). This establishes the final claim. QED

ACBC (45)

1. A ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gA.
2. C ∪. fA ⊆ C ∪. gC, C ∪. fB ⊆ C ∪. gC.

3. A ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gC.
4. A ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gA.
5. A ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gC.

6. A ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gA.
7. A ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gC.
8. A ∪. fA ⊆ C ∪. gC, C ∪. fB ⊆ C ∪. gA.
9. A ∪. fA ⊆ C ∪. gC, C ∪. fB ⊆ C ∪. gC.

10. C ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gA.
11. C ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gC.
12. C ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gA.
13. C ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gC.

14. C ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gA.
15. C ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gC.
16. C ∪. fA ⊆ C ∪. gC, C ∪. fB ⊆ C ∪. gA.

17. A ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gB.
18. A ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gA.
19. A ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gB.

20. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gA.
21. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gB.
22. A ∪. fA ⊆ C ∪. gB, B ∪. fB ⊆ C ∪. gA.
x. A ∪. fA ⊆ C ∪. gB, B ∪. fB ⊆ C ∪. gB. ≡ 18.

23. B ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gA.
24. B ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gB.
x. B ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gA. ≡ 21.
x. B ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gB. ≡ 17.

25. B ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gA.
x. B ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gB. ≡ 23.
x. B ∪. fA ⊆ C ∪. gB, B ∪. fB ⊆ C ∪. gA. ≡ 20.

26. A ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gC.
27. A ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gB.

28. A ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gC.
29. A ∪. fA ⊆ C ∪. gB, B ∪. fB ⊆ C ∪. gC.
30. A ∪. fA ⊆ C ∪. gB, C ∪. fB ⊆ C ∪. gA.
31. A ∪. fA ⊆ C ∪. gB, C ∪. fB ⊆ C ∪. gB.
32. A ∪. fA ⊆ C ∪. gB, C ∪. fB ⊆ C ∪. gC.

33. A ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gB.
x. A ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gA. ≡ 29.
x. A ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gB. ≡ 26.
34. A ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gC.
35. A ∪. fA ⊆ C ∪. gC, C ∪. fB ⊆ C ∪. gB.

36. B ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gC.
x. B ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gC. ≡ 33.
37. B ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gA.
38. B ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gB.
39. B ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gC.

40. B ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gC.
41. B ∪. fA ⊆ C ∪. gB, C ∪. fB ⊆ C ∪. gA.

x. B ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gA. ≡ 40.
x. B ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gB. ≡ 36.
42. B ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gC.
x. B ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gA. ≡ 28.
43. B ∪. fA ⊆ C ∪. gC, C ∪. fB ⊆ C ∪. gA.

x. C ∪. fA ⊆ C ∪. gA, A ∪. fB ⊆ C ∪. gB. ≡ 38.
x. C ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gA. ≡ 31.
x. C ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gB. ≡ 27.
x. C ∪. fA ⊆ C ∪. gA, B ∪. fB ⊆ C ∪. gC. ≡ 35.
44. C ∪. fA ⊆ C ∪. gA, C ∪. fB ⊆ C ∪. gB.

x. C ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gA. ≡ 41.
x. C ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gB. ≡ 37.

x. C ∪. fA ⊆ C ∪. gB, A ∪. fB ⊆ C ∪. gC. ≡ 43.
x. C ∪. fA ⊆ C ∪. gB, B ∪. fB ⊆ C ∪. gA. ≡ 30.
45. C ∪. fA ⊆ C ∪. gB, C ∪. fB ⊆ C ∪. gA.

C ∪. fA ⊆ C ∪. gC, A ∪. fB ⊆ C ∪. gB. ≡ 39.
C ∪. fA ⊆ C ∪. gC, B ∪. fB ⊆ C ∪. gA. ≡ 32.

The numbered part of this ACBC table, annotated, appears in section 3.14.

THEOREM 3.1.18. There are exactly 574 ordered pairs of clauses up to pair equivalence.

Proof: From the above tables and lemmas, we have the following counts.

SINGLE CLAUSES (DUPLICATES). 14.
AAAA. 20.
AAAB. 81.
AABA. 81.
AABB. 45.
AABC. 81.
ABAB. 36.
ABAC. 45.
ABBA. 45.
ABBC. 81.
ACBC. 45.

This adds up to a total of 574 ordered pairs up to equivalence (including the 14 duplicates). As expected, this number is a bit larger than 6561/12 = 546.75, since the overwhelmingly majority of equivalence classes have 12 elements, with a few exceptions. QED