3.2. Some Useful Lemmas.

DEFINITION 3.2.1. The standard pairing function on N is the function $P:N^2 \to N$ due (essentially) to Cantor:

$$P(n,m) = (n^2+m^2+2nm+n+3m)/2 \geq n,m.$$

It is well known that P is a bijection, and also that for all $n \geq 0$, $[0,n(n+1)/2) \subseteq P[[0,n)^2]$. In addition, P is strictly increasing in each argument.

Let $T:N^2 \to N$ be such that $T(2n,2m) = P(n,m)$, $T(2n,2m+1) = T(2n+1,2m) = T(2n+1,2m+1) = 2n+2m+2$. Then for all $n \geq 0$, $[0,n(n+1)/2) \subseteq T[([0,2n) \cap 2N)^2]$. Hence for all $n \geq 8$, every element of $[0,n^2/8)$ is realized as a value of T at even pairs from $[0,n)$.

It is clear that $T(2n,2m) \geq (n^2+2n)/2,(m^2+2m)/2 \geq 2n,2m$. Hence for $n,m \geq 2$, $T(n,m) \geq n,m$.

LEMMA 3.2.1. There exists 3-ary $f \in ELG \cap SD$ such that the following holds. Let $A \subseteq N$ be nonempty, where $fA \cap 2N \subseteq A$. Then fA is cofinite. We can also require that for all $n \geq 0$, $f(n,n,n) \in 2N$.

Proof: We define $f \in ELG \cap SD$ as follows. Let $p,q \in [2^n,2^{n+1})$, $n \geq 0$. Define $f(2^n,p,q) = \min(2^{n+1}+T(p-2^n,q-2^n),2^{n+2})$. Note that for $n \geq 8$, as p,q vary over the even elements of $[2^n,2^{n+1})$, every value in $[2^{n+1},2^{n+2})$ is realized. Also note that for all $n \geq 0$, $f(2^n,2^n,2^n) = 2^{n+1}$.

For all $n > 0$, define $f(n,n,n)$ to be the least $2^k \geq 2n$; $f(0,0,0) = 2$.

For all $n < m < r$, define $f(r,n,n) = 2r+1$, $f(r,n,m) = 2r+2$, $f(r,n,r) = 2r+3$, $f(r,m,n) = 2r+4$, $f(r,r,n) = 2r+5$. For all triples a,b,c, if f(a,b,c) has not yet been defined, define $f(a,b,c) = 2|a,b,c|+1$.

It is obvious that $f \in SD$. To see that $f \in ELG$, we need only examine the definition of $f(2^n,p,q)$, $p,q \in [2^n,2^{n+1})$, where n is sufficiently large. If $p,q \in [2^n,2^n+2^{n-1})$, then obviously $f(2^n,p,q) \geq 2^{n+1} \geq 4|2^n,p,q|/3$. If $p,q \notin [2^n,2^n+2^{n-1})$, then $f(2^n,p,q) \geq 2^{n+1}+T(p-2^n,q-2^n) \geq 2^{n+1} + 2^{n-1} \geq 5p/4,5q/4$. Also, $f(2^n,p,q) \leq 2^{n+2} \leq 2p,2q$. Therefore $f \in ELG$.

Let $A \subseteq N$ be nonempty, where $fA \cap 2N \subseteq A$. Let $f(\min(A),\min(A),\min(A)) = 2^k \geq 2$. Then $2^k \in fA \cap 2N$. Therefore $2^k \in A$.

Suppose $j \geq k$ and $2^j \in A$. Then $f(2^j,2^j,2^j) = 2^{j+1} \in fA$. We have thus established by induction that for all $j \geq k$, $2^j \in A$.

We now fix t such that $t > 8,\min(A)$, and $2^t \in A$. Then $\min(A) < 2^t < 2^{t+1}$ are all in A. Hence $\{2^{t+2},2^{t+2}+5] \subseteq fA$.

We inductively define $\alpha(0) = 6$, $\alpha(i+1) = \min((\alpha(i)^2-1)/8,2^{t+i+3})$. Note that for all sufficiently large i, $\alpha(i) = 2^{t+i+2}$.

We now prove by induction on i that for all $i \geq 0$,

1) $[2^{t+i+2},2^{t+i+2} + \alpha(i)) \subseteq fA$.

We have already established that this is true for $i = 0$. Suppose this is true for a particular $i \geq 0$. We claim that

2) $[2^{t+i+2},2^{i+t+2} + \alpha(i)) \subseteq fA$.
3) $[2^{t+i+2},2^{i+t+2} + \alpha(i)) \cap 2N \subseteq A$.
4) $[2^{t+i+3},2^{t+i+3} + \alpha(i+1)) \subseteq f(([2^{t+i+2},2^{t+i+2} + \alpha(i)) \cap 2N)^2) \subseteq fA$.

2) is the induction hypothesis. 3) follows from 2) and $fA \cap 2N \subseteq A$.

For 4), let $x \in [2^{t+i+3},2^{t+i+3} + \alpha(i+1)) \subseteq [2^{t+i+3},2^{t+i+4})$. Then $0 \leq x-2^{t+i+3} < \alpha(i+1) \leq (\alpha(i)^2-1)/8$. By the choice of T, let $a,b < \alpha(i)$, $T(a,b) = x-2^{t+i+3}$, where a,b are even. Let $p = 2^{t+i+2} + a$, $q = 2^{t+i+2} + b$. Then $p,q \in [2^{t+i+2},2^{t+i+2} + \alpha(i))$, p,q are even, and $f(2^{t+2i+2},p,q) = x$.

This establishes that $[2^{t+i+3},2^{t+i+3} + \alpha(i+1)) \subseteq f[([2^{t+i+2},2^{t+i+1} + \alpha(i)) \cap 2N)^2]$. $f[([2^{t+i+2},2^{t+i+1} + \alpha(i)) \cap 2N)^2] \subseteq fA$ is immediate from $[2^{t+i+2},2^{i+t+2} + \alpha(i)) \cap 2N \subseteq A$.

This concludes the inductive argument for 1). Since for sufficiently large i, $\alpha(i) = 2^{t+i+2}$, we see that fA is cofinite. QED

We will need the following technical refinement of Lemma 3.2.1.

LEMMA 3.2.2. There exists 4-ary $g \in ELG \cap SD$ such that the following holds. Let $A \subseteq N$ have at least two elements, where $(\forall n \in gA \cap 2N)(4n+3 \in gA \rightarrow n \in A)$. Then gA is cofinite. We can also require that for all $n \in N$, $g(n,n,n,n) \in 2N$.

Proof: Let $f:N^3 \rightarrow N$ be as given by Lemma 3.2.1. We define $g:N^4 \rightarrow N$ as follows. Let $x \in N^3$. If $n = |x|$ then define $g(n,x) = f(x)$. If $n < |x|$ then define $g(n,x) = 4f(x)+3$. If $n > |x|$ then define $g(n,x) = 2n+1$. Note that $g(n,n,n,n) = f(n,n,n) \in 2N$. Also, if $n < |m,r,s|$ then $g(n,m,r,s) \geq f(m,r,s) > m,r,s$, and if $n > m,r,s$, then $g(n,m,r,s) > n,m,r,s$. Hence $g \in ELG \cap SD$.

Let A be as given. Let $A' = A\backslash\{\min(A)\}$. Then A' is nonempty. Let $n \in fA' \cap 2N$. Let $n = f(x)$, $x \in A'^3$. Hence $4n+3 \in gA$ using min(A) as the first argument for g. Therefore $n \in A$, and so $n \in A'$.

We have thus shown that $fA' \cap 2N \subseteq A'$. By Lemma 3.2.1, fA' is cofinite. Hence gA is cofinite. QED

We will need a refinement of Lemma 3.2.1 in a different direction (Lemma 3.2.4).

LEMMA 3.2.3. Let $f \in ELG \cap SD$ have arity p. There exists $g,h_1,h_2 \in ELG \cap SD$, with arities 2p,1,1 respectively, such that $f(x_1,\ldots,x_p) = g(h_1(x_1),\ldots,h_1(x_p),h_2(x_1),\ldots,h_2(x_p))$ holds, with finitely many exceptional p-tuples. We can also require that $rng(h_1),rng(h_2) \subseteq 2N$, and each $g(n,\ldots,n)$ is even.

Proof: Let f,p be as given. Let $c,d > 1$ be rational constants such that

$$c|x| \leq f(x) \leq d|x|$$

holds with finitely many exceptions. Let t be sufficiently large relative to c,d. We can assume that $1 < c < 2 < d$.

We first define $h_1,h_2:[t,\infty) \rightarrow N$ by

$$h_1(x) = \text{the first integer} > c^{1/3}x \text{ that is divisible by 4.}$$
$$h_2(x) = h_1(x) + 4(x \bmod 8) + 4.$$

To see that $h(x) = (h_1(x),h_2(x))$ is one-one on $[t,\infty)$, suppose $h_1(x) = h_1(y)$ and $h_2(x) = h_2(y)$ and $x < y$. By subtraction, $4(x \bmod 8) + 4 = 4(y \bmod 8) + 4$, $x \equiv y \bmod 8$, and so $y \geq x+8$. Hence the first integer $> c^{1/3}y$ is at least the first integer $> c^{1/3}x$, plus 8. Hence $h_1(x) \neq h_1(y)$.

Extend h_1,h_2 on $[0,t)$ by

$$h_1(x) = h_2(x) = 2x+2.$$

Note that

$$c^{1/3}x \leq h_1(x),h_2(x) \leq 2x+2.$$

Hence $h_1,h_2 \in \text{ELG} \cap \text{SD}$, $\text{rng}(h_1) \cup \text{rng}(h_2) \subseteq 2N$, and h is one-one. Also $h_1(x) \leq h_1(x+1)$, and $h_1(x) < h_2(x) \leq h_1(x) + 36$.

We define $g:N^{2p} \to N$ as follows.

case 1. $(y_1,z_1),\ldots,(y_p,z_p) \in \text{rng}(h)$, and $|y_1,\ldots,y_p,z_1,\ldots,z_p| > ct$. Set $g(y_1,\ldots,y_p,z_1,\ldots,z_p) = f(h^{-1}(y_1,z_1),\ldots,h^{-1}(y_p,z_p))$.

case 2. Otherwise. Set $g(y_1,\ldots,y_p,z_1,\ldots,z_p) = 2|y_1,\ldots,y_p,z_1,\ldots,z_p|+2$.

We claim that $g \in \text{ELG} \cap \text{SD}$. To see this, note that g restricted to case 2 lies in ELG ∩ SD. So it remains to consider case 1.

Let $h(x_1) = (y_1,z_1),\ldots,h(x_p) = (y_p,z_p)$. Then for all i,

$$h_1(x_i) = y_i,\ h_2(x_i) = z_i.$$
$$y_i,z_i \geq x_i.$$

Also let j be such that x_j is largest. Then $x_j = |y_1,\ldots,y_j| \geq t$, and so $x_j \geq |y_1,\ldots,y_p,z_1,\ldots,z_p| - 36$. Hence

$$x_j \geq c^{-1/3}|y_j,z_j| \geq c^{-1/2}|y_1,\ldots,y_p,z_1,\ldots,z_p|.$$

$$g(y_1,\ldots,y_p,z_1,\ldots,z_p) = f(x_1,\ldots,x_p) \leq d|x_1,\ldots,x_p| \leq d|y_1,\ldots,y_p,z_1,\ldots,z_p|.$$

$$g(y_1,\ldots,y_p,z_1,\ldots,z_p) = f(x_1,\ldots,x_p) \geq c|x_1,\ldots,x_p| = cx_j \geq cc^{-1/2}|y_1,\ldots,y_p,z_1,\ldots,z_p| \geq c^{1/2}|y_1,\ldots,y_p,z_1,\ldots,z_p|.$$

Hence $g \in \text{ELG} \cap \text{SD}$. Note that the case $g(n,\ldots,n)$ must lie in case 2. Hence $g(n,\ldots,n) \in 2N$.

Finally,

$$f(x_1,...,x_p) = g(h_1(x_1),...,h_1(x_p),h_2(x_1),...,h_2(x_p))$$

holds according to case 1. The only exceptions are if $|h_1(x_1),...,h_1(x_p),h_2(x_1),...,h_2(x_p)| \leq ct$. But that is at most finitely many exceptions. QED

LEMMA 3.2.4. There exists a 8-ary $F \in ELG \cap SD$ such that the following holds. Let $A \subseteq N$ be nonempty, where $F(FA \cap 2N) \cap 2N \subseteq A$. Then FA is cofinite.

Proof: Let $f:N^3 \rightarrow N$ be as given by Lemma 3.2.1. By Lemma 3.2.3, let $g,h_1,h_2 \in ELG \cap SD$, with arities 6,1,1 respectively, such that

$$f(x,y,z) = g(h_1(x),h_1(y),h_1(z),h_2(x),h_2(y),h_2(z))$$

with finitely many exceptions, where $rng(h_1),rng(h_2) \subseteq 2N$, and each $g(n,...,n) \in 2N$.

We now define $F:N^8 \rightarrow N$ by cases.

case 1. $x_1 = x_2 = |x_3,...,x_8|$. Set $F(x_1,...,x_8) = g(x_3,...,x_8)$.

case 2. $x_1 = x_2 < x_3 = ... = x_8$. Set $F(x_1,...,x_8) = h_1(x_3)$.

case 3. $x_1 < x_2 < x_3 = ... = x_8$. Set $F(x_1,...,x_8) = h_2(x_3)$.

case 4. $x_2 < x_1 < |x_3,x_4,x_5| = |x_1,...,x_8|$. Set $F(x_1,...,x_8) = f(x_3,x_4,x_5)$.

case 5. Otherwise. Set $F(x_1,...,x_8) = 2|x_1,...,x_8|+1$.

It is obvious that $F \in ELG \cap SD$.

Assume $F(FA \cap 2N) \cap 2N \subseteq A$, where A is nonempty. Let $n \in A$. Then $F(n,...,n) \in 2N$, and we can keep applying F to diagonals, thereby obtaining an infinite subset of $A \cap 2N$.

Let A' be the tail of A whose least element is greater than exactly two elements of A.

We claim that $fA' \subseteq F(FA' \cap 2N)$. To see this, let $n < m$ be the first two elements of A. Then by cases 2 and 3 above, for all $r \in A'$, $h_1(r), h_2(r) \in FA \cap 2N$. Let $x,y,z \in A'$. Now $f(x,y,z) = g(h_1(x),h_1(y),h_1(z),h_2(x),h_2(y),h_2(z)) = F(p,p,h_1(x),h_1(y),h_1(z),h_2(x),h_2(y),h_2(z)) \in F(FA \cap 2N)$, where $p = |h_1(x),h_1(y),h_1(z),h_2(x),h_2(y),h_2(z)|$.

In particular, $fA' \cap 2N \subseteq F(FA \cap 2N) \cap 2N \subseteq A$. Since f is strictly dominating, $fA' \cap 2N \subseteq A'$. By Lemma 3.2.1, fA' is cofinite.

Clearly $fA' \subseteq FA$ by case 4. Hence FA is cofinite. QED

Let $f_1,\ldots,f_k$ be indeterminate functions from EVSD. We consider the class of $f_1,\ldots,f_k$,A-terms defined as follows.

i. A is an $f_1,\ldots,f_k$,A-term.
ii. If s,t are $f_1,\ldots,f_k$,A-terms, then $s \cup t$ is an $f_1,\ldots,f_k$,A-term.
iii. If s is an $f_1,\ldots,f_k$,A-term, then each $f_i s$ is an $f_1,\ldots,f_k$,A-term.

LEMMA 3.2.5. Let $k \geq 1$, $f_1,\ldots,f_k \in$ EVSD, and $t_1,\ldots,t_r$ be $f_1,\ldots,f_k$,A-terms. There exists $A \in$ INF such that each $A \cap t_i = \emptyset$. We can require that min(A) be any given sufficiently large integer.

Proof: Let $f_1,\ldots,f_k \in$ EVSD. Write each $t_i = t_i(f_1,\ldots,f_k,A)$. Let n be sufficiently large. We define integers $n_0 < n_1 < \ldots$ as follows. Let $n_0 = n$. Suppose n_j has been defined, $j \geq 0$. Let n_{j+1} to be such that

n_{j+1} is greater than n_j and all elements of each $t_i(f_1,\ldots,f_k,\{n_0,\ldots,n_j\})$.

Take $A = \{n_j: j \geq 0\}$. QED