3.3. Single Clauses (duplicates).

In this section we handle the relatively easy case of ordered pairs α,β of clauses, where $\alpha = \beta$. We these duplicate ordered pairs as single clauses, α.

As we shall see, several single clauses have ¬NON, and so any ordered pair of clauses, at least one of which is such a clause, also has ¬NON, and does not have to be further considered. This will allow us to cut down significantly on the number of pairs of clauses that have to be considered in sections 3.4 - 3.13.

By Lemma 3.1.5, we see that every clause is equivalent to a clause whose inner signature is AA or AB.

Here are what we call the AA and AB tables, together with the outcomes of our five attributes, INF, AL, ALF, FIN, NON, introduced in section 3.1. These entries are justified by the Lemmas that follow.

AA

1. A ∪. fA ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2. A ∪. fA ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3. A ∪. fA ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
4. B ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. B ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. B ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
7. C ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
8. C ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
9. C ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. B ∪. fA ⊆ B ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5. B ∪. fA ⊆ B ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6. B ∪. fA ⊆ B ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
7. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
8. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
9. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

According to the procedure specified at the beginning of this Chapter, in order to validate TEMP 3, we use EVSD for the positive entries with attribute INF (other than the Exotic Case). Otherwise, we will always use ELG.

The following pertains to AA 1-3. Note that in the statement of Lemma 3.3.1, we use X as an unknown representing A,B, or C. We will make use of this convention throughout this Chapter.

LEMMA 3.3.1. $A \cup. fA \subseteq A \cup. gX$ has $\neg$NON.

Proof: Define $f,g \in$ ELG as follows. Let $f(n) = 2n$, $g(n) = 2n+1$. Let $A \cup. fA \subseteq A \cup. gX$, where A,X are nonempty. Let $n \in A$. Then $2n \in fA$, $2n \in A$. This contradicts $A \cap fA = \emptyset$. QED

The following pertains to AA 4-9.

LEMMA 3.3.2. $X \cup. fA \subseteq A \cup. gY$ has $\neg$INF, $\neg$FIN.

Proof: Let f be as given by Lemma 3.2.1. Let $g \in$ ELG be defined by $g(n) = 2n+1$. Suppose $X \cup. fA \subseteq A \cup. gY$, where X,A,Y are nonempty. Then $fA \cap 2N \subseteq A$. Hence fA is cofinite. Since $X \cap fA = \emptyset$, we have that A is infinite and X is finite. This establishes that $\neg$INF, $\neg$FIN. QED

LEMMA 3.3.3. Let $g \in$ EVSD. Let n be sufficiently large. For all $S \subseteq [n,\infty)$, there exists a unique $A \subseteq S \subseteq A \cup. gA$. Furthermore, if S is infinite then A is infinite.

Proof: This is a variant of the Complementation Theorem from Section 1.3. Since n is sufficiently large, g is strictly dominating at all tuples x with $|x| \geq n$.

We define $A \subseteq S$ by induction on $k \in S$. Suppose membership in A for all $i \in S \cap [n,k)$ has been determined, where $k \in S$. We put k in A if and only if k is not yet a value of g at arguments from A. Note that if k is not yet a value of g at arguments from A, then k will never become a value of g at arguments from A. Hence $S \subseteq A \cup. gA$. It is clear from this inclusion that if S is infinite, then A is infinite.

For uniqueness, let $A \subseteq S \subseteq A \cup. gA$ and $B \subseteq S \subseteq B \cup. gB$. Let k be least such that $k \in A \leftrightarrow k \notin B$. Obviously, $k \in S$ and

$$k \in A \leftrightarrow k \notin gA.$$
$$k \in B \leftrightarrow k \notin gB.$$

Since g is strictly dominating on $[n,\infty)$, $A,B \subseteq [n,\infty)$, and $k \geq n$, we see that

$$k \in gA \leftrightarrow k \in g(A \cap [0,k)).$$
$$k \in gB \leftrightarrow k \in g(B \cap [0,k)).$$

Hence

$$k \in A \leftrightarrow k \notin g(A \cap [0,k)).$$
$$k \in B \leftrightarrow k \notin g(B \cap [0,k)).$$

Since $A \cap [0,k) = B \cap [0,k)$, we have

$$k \in A \leftrightarrow k \in B$$

contradicting the choice of k. QED

The following pertains to AA 4.

LEMMA 3.3.4. $B \cup. fA \subseteq A \cup. gA$ has AL.

Proof: Let $f,g \in$ ELG and $p > 0$. Let n be sufficiently large. Then $[n,n+p] \notin f[[n,\infty)] \cup g[[n,\infty)]$. By Lemma 3.3.3, let $A \subseteq [n,\infty) \subseteq A \cup. gA$. Then $[n,n+p] \subseteq A$. Let $B = [n,n+p]$. QED

The following pertains to AA 5.

LEMMA 3.3.5. $B \cup. fA \subseteq A \cup. gB$ has AL.

Proof: Let $f,g \in$ ELG and $p > 0$. Let $B = [n,n+p]$, where n is sufficiently large. Let $A = [n,\infty)\backslash gB$. Since $B \cup fA \subseteq [n,\infty)$, we have $B \cup fA \subseteq A \cup gB$. Also $B \cap f([n,\infty)) = \emptyset$. QED

The following pertains to AA 4 - 9.

LEMMA 3.3.6. $X \cup. fA \subseteq A \cup. gY$ has AL, provided $X \in \{B,C\}$.

Proof: Let $f,g \in$ ELG and $p > 0$. By Lemma 3.3.4, let A,B have at least p elements, where $B \cup. fA \subseteq A \cup. gA$. By setting $C = B$, we see that AA 7 has AL.

By Lemma 3.3.5, let A,B have at least p elements, where $B \cup. fA \subseteq A \cup. gB$. By setting $C = B$, we see that AA 6,8,9 have AL. QED

The following pertains to AB 4 - 6.

LEMMA 3.3.7. B ∪. fA ⊆ B ∪. gX has ¬NON.

Proof: Define f,g ∈ ELG by f(n) = 2n, g(n) = 2n+1. Let B ∪. fA ⊆ B ∪. gX, where A,B,X are nonempty. Let n ∈ A. Then 2n ∈ fA, 2n ∈ B. This contradicts B ∩ fA = ∅. QED

The following pertains to AB 1,3,7,9.

LEMMA 3.3.8. X ∪. fA ⊆ B ∪. gY has INF, ALF, provided X,Y ∈ {A,C}, even for EVSD.

Proof: Let f,g ∈ EVSD. By Theorem 3.2.5, let A be infinite, where A ∩ fA = A ∩ gA = ∅. Let C = A and B = (A ∪ fA)\gA. Then A ⊆ B, and so A,B,C are infinite. This establishes INF.

For ALF, let p > 0. Let A be the first p elements of the above A, where A ∩ fA = A ∩ gA = ∅. Let C = A and B = (A ∪ fA)\gA. Then A ⊆ B, and so |B| ≥ p and A,B,C are finite. QED

The following pertains to AB 2,8.

LEMMA 3.3.9. X ∪. fA ⊆ B ∪. gB has INF, ALF, provided X ∈ {A,C}, even for EVSD.

Proof: Let f,g ∈ EVSD and n be sufficiently large. By Theorem 3.2.5, let A ⊆ [n,∞) be infinite, where A ∩ fA = A ∩ gA = ∅. By Lemma 3.3.3, let B be unique such that B ⊆ A ∪ fA ⊆ B ∪. gB. Let C = A. Since A ∪ fA is infinite, B is infinite. This establishes INF.

Now let p > 0 be given. Let A be the first p elements of the above A. Then A ∩ fA = A ∩ gA = ∅. Let B be the unique B ⊆ A ∪ fA such that A ∪ fA ⊆ B ∪. gB. Let C = A. Since A ∩ gB = ∅, we have A ⊆ B. This establishes ALF. QED

The information contained in these Lemmas is sufficient to justify all determinations made on the AA and AB tables, using the obvious implications

ALF → AL → NON.
ALF → FIN → NON.
INF → AL → NON.

and contrapositives.

Lemma 3.3.7 is particularly useful. It allows us to remove a large number of pairs of clauses in sections 3.4 - 3.13 (e.g., see the reduced AA table at the beginning of section 3.4). Also, it allows us to automatically annotate a very large number of entries in the annotated tables of section 3.14.

We now illustrate a difference between ELG and SD with respect to AL. We have the following, in contrast to Lemma 3.3.4.

THEOREM 3.3.10. There exist $f,g \in SD$ such that the following holds. Let $B \cup. fA \subseteq A \cup. gA$. If A is nonempty then B has at most one element. In particular, this clause for SD has attribute ¬AL, and this clause for ELG has attribute AL (Lemma 3.3.4).

Proof: For $n < m$, let $f(n,n) = n+1$, $f(n,m) = m+1$, $f(m,n) = m+2$. Let $g(n) = 2n+3$. Let $B \cup. fA \subseteq A \cup. gA$, where A is nonempty. Let $n = \min(A)$. Then $n+1 \in A \cup gA$, $n+1 \notin gA$, $n+1 \in A$.

We claim that $[n+1,\infty) \subseteq fA$. Since $n \in A$, clearly $n+1 \in fA$. Hence $n+1 \in A \cup gA$. Now $n+1 \in gA$ is impossible since $n = \min(A)$. Hence $n+1 \in A$, $n+2 \in fA$.

Now let $[n+1,m] \subseteq fA$, $m \geq n+2$. To establish the claim, it suffices to prove that $m+1 \in fA$. Now $m \in fA$, $m \in A \cup gA$. If $m \in A$ then $m+1 \in fA$. So it suffices to assume that $m \in gA$. Hence m is odd. Also $m-1 \in fA$, $m-1 \in A \cup gA$. Since $m-1$ is even, $m-1 \in A$. Let $r < m-1$, $r \in A$. Then $f(m-1,r) = m+1 \in fA$.

We have thus established that $[n+1,\infty) \subseteq fA$.

Now let $r \in B$. By the above claim, $r \leq n$, $r \in A \cup gA$, $r \in A$, $r = n$. Hence B has exactly one element. QED