3.6. AABA.

Recall the reduced AA table from section 3.4.

REDUCED AA

1. B ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
2. B ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
3. B ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
4. C ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. C ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. C ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

Recall the reduced AB table from section 3.5.

REDUCED AB

1. A ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
2. A ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
3. A ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.
4. C ∪. fA ⊆ B ∪. gA. INF. AL. ALF. FIN. NON.
5. C ∪. fA ⊆ B ∪. gB. INF. AL. ALF. FIN. NON.
6. C ∪. fA ⊆ B ∪. gC. INF. AL. ALF. FIN. NON.

The reduced BA table is obtained from the reduced AB table by switching A,B. We use 1′-6′ to avoid any confusion.

REDUCED BA

1′. B ∪. fB ⊆ A ∪. gB. INF. AL. ALF. FIN. NON.
2′. B ∪. fB ⊆ A ∪. gA. INF. AL. ALF. FIN. NON.
3′. B ∪. fB ⊆ A ∪. gC. INF. AL. ALF. FIN. NON.
4′. C ∪. fB ⊆ A ∪. gB. INF. AL. ALF. FIN. NON.
5′. C ∪. fB ⊆ A ∪. gA. INF. AL. ALF. FIN. NON.
6′. C ∪. fB ⊆ A ∪. gC. INF. AL. ALF. FIN. NON.

We consider all 36 pairs, arranged in cases according to the first clause of the ordered pair.

The status of all of our proposition attributes are determined by the reduced AA table except AL and NON. Thus, we need only obtain the status of AL and NON.

part 1. B ∪. fA ⊆ A ∪. gA.

1,1′. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. NON.

1,2′. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
1,3′. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
1,4′. B ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. NON.
1,5′. B ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
1,6′. B ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. NON.

LEMMA 3.6.1. There exists g ∈ ELG ∩ SD such that the following holds. Suppose A ∪ gB is cofinite and A ∩ gA = ∅. Then A ⊆ B. We can require that rng(g) ⊆ 2N+1. Furthermore, we can require that for all X and n, 4n+3 ∈ gX ↔ n ∈ X.

Proof: Define g ∈ ELG ∩ SD as follows. For all m > n, define

$$g(n,4m^2+4n+1) = 16m^2+4n+1.$$

For all other pairs p,q, define

$$g(p,q) = 4|p,q|+3.$$

Let A ∪ gB be cofinite and A ∩ gB = ∅. Let n ∈ A\B. We derive a contradiction.

Note that the last two requirements on g hold.

We first claim that

$$m > n \rightarrow 4m^2+4n+1 \notin gB.$$

To see this, let m > n, $4m^2+4n+1 \in gB$. Note that $4m^2+4n+1 \equiv 1 \bmod 4$. Hence for some n′,m′ ∈ B, m′ > n′, we have

$$4m^2+4n+1 = g(n',m') = 16m'^2+4n'+1.$$

Since n ∉ B and n′ ∈ B, we have n ≠ n′. Also

$$16m'^2 - 4m^2 = 4n - 4n'.$$
$$4m'^2 - m^2 = n - n'.$$
$$(2m' - m)(2m' + m) = n - n'.$$
$$2m' - m \neq 0.$$
$$2m' + m > 2n' + n.$$
$$2n' + n < |(2m' - m)(2m' + m)| = |n - n'| \leq n + n'.$$

$$n' < 0.$$

Now fix $m > n$, where $4m^2+4n+1$, $16m^2+4n+1 \in A \cup gB$. By the first claim applied to m and to 2m, we have

$$4m^2+4n+1,\ 16m^2+4n+1 \notin gB.$$
$$4m^2+4n+1,\ 16m^2+4n+1 \in A.$$
$$n \in A.$$
$$g(n,4m^2+4n+1) = 16m^2+4n+1 \in gA.$$

This contradicts $A \cap gA = \emptyset$. QED

LEMMA 3.6.2. $B \cup. fA \subseteq X \cup. gX$, $fB \subseteq X \cup. gB$ has ¬AL.

Proof: Let f be given by Lemma 3.2.2. Let g be as given by Lemma 3.6.1. Let $B \cup. fA \subseteq X \cup. gX$, $fB \subseteq X \cup. gB$, where A,B,C have at least two elements. We now use Lemma 3.2.2 to show that fB is cofinite.

Let $n \in fB \cap 2N$, $4n+3 \in fB$. Then $n \in X$, $4n+3 \in gX$, $4n+3 \notin X$. Since $4n+3 \in fB$, we have $4n+3 \in gB$. Hence $n \in B$. We have thus established that $(\forall n \in fB \cap 2N)(4n+3 \in fB \rightarrow n \in B)$. By Lemma 3.2.2, fB is cofinite.

We have thus established that $X \cup gB$ is cofinite and $X \cap gX = \emptyset$. By Lemma 3.6.1, $X \subseteq B$. By Lemma 3.2.2, fA has an even element 2r. Hence $2r \in X$, $2r \in B$. This contradicts $B \cap fA = \emptyset$. QED

LEMMA 3.6.3. 1,1′, 1,4′ have ¬AL.

Proof: By Lemma 3.6.2, setting X = A. QED

The following pertains to 1,6′.

LEMMA 3.6.4. $B \cup. fA \subseteq A \cup. gA$, $C \cup. fB \subseteq A \cup. gC$ has ¬AL.

Proof: Define $f,g \in ELG$ as follows. For all $n < m$, let $f(n,n) = 2n+2$, $f(n,m) = f(m,n) = 4m+5$, $g(n) = 2n+1$. Let $B \cup. fA \subseteq A \cup. gA$, $C \cup. fB \subseteq A \cup. gC$, where A,B,C have at least two elements. Let $n < m$ be from B.

Clearly $2m+2, 4m+5 \in fB$, $2m+2 \notin C$, $4m+5 \notin gC$, $4m+5 \in A$, $4m+5 \notin gA$, $2m+2 \notin A$, $2m+2 \in gC$. This is impossible since g is odd valued. QED

The following pertains to 1,2′, 1,5′.

LEMMA 3.6.5. X ∪. fA ⊆ A ∪. gA, Y ∪. fB ⊆ A ∪. gA has AL, provided X,Y ∈ {B,C}.

Proof: Let f,g ∈ ELG and p > 0. Let B = C = [n,n+p], where n is sufficiently large. By Lemma 3.3.3, let A be unique such that A ⊆ [n,∞) ⊆ A ∪. gA.

Obviously X ∩ fA = X ∩ gA = A ∩ gA = Y ∩ fB = ∅. Hence B,C ⊆ A. Also fA,fB ⊆ [n,∞) ⊆ A ∪ gA. QED

The following pertains to 1,3'.

LEMMA 3.6.6. B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gC has AL.

Proof: By Lemma 3.6.5, B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gA has AL. Replace C by A in the cited pair. QED

LEMMA 3.6.7. X ∪. fA ⊆ A ∪. gA, Y ∪. fZ ⊆ A ∪. gW has NON, provided X,Y,Z,W ∈ {B,C}.

Proof: Let f,g ∈ ELG. Let n be sufficiently large.

case 1. f(n,...,n) = g(n,...,n). Let A = B = C = {n}.

case 2. f(n,...,n) ≠ g(n,...,n). Let B = C = {n}. By Lemma 3.3.3, let A be unique such that A ⊆ [f(n,...,n),∞) ∪ {n} ⊆ A ∪. gA.

In case 1, both inclusions have the same left and right sides, and are easily verified.

We assume case 2 holds. Obviously B ∩ fA = B ∩ fB = A ∩ gA = ∅. Also n ∈ A, and hence X ⊆ A and Y ⊆ A. Since g(n,...,n) ∈ gA, we have g(n,...,n) ∉ A. Hence A ∩ gB = A ∩ gC = ∅.

We have thus shown that X ∩ fA = A ∩ gA = Y ∩ fZ = A ∩ gW = ∅.

Note that f(n,...,n) ∉ gA. To see this, let f(n,...,n) = $g(b_1,...,b_r)$, $b_1,...,b_r$ ∈ A. Clearly not every b_i is n. Hence some b_i is at least f(n,...,n). This is a contradiction.

Since f(n,...,n) ∉ gA, we see that f(n,...,n) ∈ A. Hence fZ ⊆ A. Also fA ⊆ [f(n,...,n),∞) ⊆ A ∪ gA. QED

LEMMA 3.6.8. 1,1′, 1,4′, 1,6′ have NON.

Proof: Immediate from Lemma 3.6.7. QED

part 2. B ∪. fA ⊆ A ∪. gB.

2,1′. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
2,2′. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,3′. B ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
2,4′. B ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
2,5′. B ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
2,6′. B ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

The following pertains to 2,1′, 2,3′, 2,4′, 2,6′.

LEMMA 3.6.9. X ∪. fA ⊆ A ∪. gY, Z ∪. fB ⊆ A ∪. gW has AL, provided X,Y,Z,W ∈ {B,C}.

Proof: Let f,g ∈ ELG and p > 0. Let B = C = [n,n+p], where n is sufficiently large. Let A = [n,∞)\gB. Then A is infinite.

Clearly B ∩ fA = C ∩ fA = B ∩ fB = C ∩ fB = A ∩ gB = A ∩ gC = B ∩ gB = C ∩ gB = ∅. Hence B,C ⊆ A. Also fA,fB ⊆ [n,∞) ⊆ A ∪ gB = A ∪ gC. QED

LEMMA 3.6.10. fA ⊆ A ∪. gB, B ∩ fA = A ∩ gA = ∅ has ¬NON.

Proof: Let f be given by Lemma 3.2.1. Define g ∈ ELG by g(n) = 2n+1. Let fA ⊆ A ∪. gB, B ∩ fA = A ∩ gA = ∅, where A,B,C are nonempty.

Obviously fA ∩ 2N ⊆ A. By Lemma 3.2.1, fA is cofinite. Since A ∩ gA = ∅, we see that A is not cofinite. Since fA ⊆ A ∪ gB and fA is cofinite, we see that gB is infinite. Hence B is infinite. This contradicts B ∩ fA = ∅. QED

LEMMA 3.6.11. 2,2′, 2,5′ have ¬NON.

Proof: Immediate from Lemma 3.6.10. QED

part 3. B ∪. fA ⊆ A ∪. gC.

3.1′. B ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,2′. B ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,3′. B ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,4′. B ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
3,5′. B ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
3,6′. B ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.6.12. 3,1′, 3,3′, 3,4′, 3,6′ have AL.

Proof: By Lemma 3.6.9. QED

The following pertains to 3,2′.

LEMMA 3.6.13. B ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gA has AL.

Proof: By Lemma 3.6.5, B ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gA has AL. Replace C by A in the cited ordered pair. QED

The following pertains to 3,5′.

LEMMA 3.6.14. B ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gA has ¬NON.

Proof: For n < m, define f(n,n,n) = 2n+2, f(n,m,m) = 4m+5, f(n,n,m) = 2m+1, f(m,n,n) = 8m+9. Define f(a,b,c) = 2|a,b,c|+1 for all other triples a,b,c. Define g(n) = 4n+5. Obviously f,g ∈ ELG.

Let B ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gA, where A,B,C ⊆ N are nonempty. Let n ∈ B. Then n ∈ A ∪ gC.

case 1. n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A, 8n+13 ∈ fA,gA, 8n+13 ∉ A, 8n+13 ∈ gC, 2n+2 ∈ C, 2n+2 ∈ fB. This contradicts C ∩ fB = ∅.

case 2. n ∈ gC. Let n = 4m+5, m ∈ C. Then m ∈ A ∪ gA.

case 2a. m ∈ A. Then 2m+2 ∈ fA, 2m+2 ∈ A, 4m+5 ∈ fA, 4m+5 = n ∈ B. This contradicts B ∩ fA = ∅.

case 2b. m ∈ gA. Let m = 4r+5, r ∈ A. Then 2r+2 ∈ fA, 2r+2 ∈ A. Since n = 4m+5 and m = 4r+5, we have n = 16r+25. Hence n = f(2r+2,r,r) ∈ fA, n ∈ B. This contradicts B ∩ fA = ∅. QED

part 4. C ∪. fA ⊆ A ∪. gA.

4,1'. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gB. ¬INF. ¬AL. ¬ALF. ¬FIN. NON.
4,2'. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,3'. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. NON.
4,4'. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,5'. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,6'. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gC. ¬INF. ¬AL. ¬ALF. ¬FIN. NON.

The following pertains to 4,1'.

LEMMA 3.6.15. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gB has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = f(m,n) = 4m+1, g(n) = 2n+1. Let fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gB, where A,B,C have at least two elements. Let n < m be from B.

Note that 2m ∈ fB, 2m ∈ A, 2m ∉ B, 4m+1 ∉ gB, 4m+1 ∈ fB, 4m+1 ∈ A, 4m+1 ∈ gA. This contradicts A ∩ gA = ∅. QED

LEMMA 3.6.16. 4,2', 4,5' have AL.

Proof: By Lemma 3.6.5. QED

The following pertains to 4,3'.

LEMMA 3.6.17. C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gC has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = 4m, f(m,n) = 8m+1, g(n) = 2n+1. Let C ∪. fA ⊆ A ∪. gA, B ∪. fB ⊆ A ∪. gC, where A,B,C have at least two elements. Let n < m be from B.

Clearly 2m ∈ fB, 2m ∈ A, 2m ∉ B, 4m ∈ fB, 4m ∈ A, 4m ∉ B, 8m+1 ∈ gA, 8m+1 ∉ A, 8m+1 ∈ fB, 8m+1 ∈ gC, 4m ∈ C, 4m ∈ fA. This contradicts C ∩ fA = ∅. QED

The following pertains to 4,4'.

LEMMA 3.6.18. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gB has AL.

Proof: From the reduced AA table, C ∪. fA ⊆ A ∪. gA has AL. Replace B by A in the cited ordered pair. QED

The following pertains to 4,6'.

LEMMA 3.6.19. C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gC has ¬AL.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n, f(n,m) = f(m,n) = 4m+1, g(n) = 2n+1. Let C ∪. fA ⊆ A ∪. gA, C ∪. fB ⊆ A ∪. gC, where A,B,C have at least two elements. Let n < m be from B.

Clearly 2m ∈ fB, 2m ∈ A, 4m+1 ∈ gA, 4m+1 ∉ A, 4m+1 ∈ fB, 4m+1 ∈ gC, 2m ∈ C. This contradicts C ∩ fB = ∅. QED

LEMMA 3.6.20. 4,1', 4,3', 4,6' have NON.

Proof: By Lemma 3.6.7, X ∪. fA ⊆ A ∪. gA, Y ∪. fZ ⊆ A ∪. gW has NON, provided X,Y,Z,W ∈ {B,C}. QED

part 5. C ∪. fA ⊆ A ∪. gB.

5,1'. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,2'. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
5,3'. C ∪. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,4'. C ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,5'. C ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,6'. C ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.6.21. 5,1', 5,3', 5,4', 5,6' have AL.

Proof: By Lemma 3.6.9, X ∪. fA ⊆ A ∪. gY, Z ∪. fB ⊆ A ∪. gW has AL, provided X,Y,Z,W ∈ {B,C}. QED

The following pertains to 5,5'.

LEMMA 3.6.22. C ∪. fA ⊆ A ∪. gB, C ∪. fB ⊆ A ∪. gA has AL.

Proof: From the reduced table for AA, we see that C ∪. fA ⊆ A ∪. gA has AL. In the cited ordered pair, replace B by A. QED

The following pertains to 5,2'.

LEMMA 3.6.23. fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gA has ¬NON.

Proof: Define f,g ∈ ELG as follows. For all n < m, let f(n,n) = 2n+2, f(m,n) = f(n,m) = 2m+1, g(n) = 2n+1. Let fA ⊆ A ∪. gB, B ∪. fB ⊆ A ∪. gA, where A,B are nonempty.

Let n ∈ A. Then 2n+2 ∈ fA, 2n+2 ∈ A, 4n+5 ∈ gA, 4n+5 ∉ A. Since n < 2n+2 are from A, we have 4n+5 ∈ fA, 4n+5 ∈ gB, 2n+2 ∈ B, 4n+6 ∈ fB, 4n+6 ∈ A. Since n < 4n+6 are from A, we have 8n+13 ∈ fA, 8n+13 ∈ gA, 8n+13 ∉ A, 8n+13 ∈ gB, 4n+6 ∈ B. This contradicts B ∩ fB = ∅. QED

part 6. C ∪. fA ⊆ A ∪. gC.

6,1'. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6,2'. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,3'. C ∪. fA ⊆ A ∪. gC, B ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
6,4'. C ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6,5'. C ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gA. ¬INF. ¬AL. ¬ALF. ¬FIN. ¬NON.
6,6'. C ∪. fA ⊆ A ∪. gC, C ∪. fB ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.6.24. 6,1', 6,3', 6,4', 6,6' have AL.

Proof: By Lemma 3.6.9, X ∪. fA ⊆ A ∪. gY, Z ∪. fB ⊆ A ∪. gW has AL, provided X,Y,Z,W ∈ {B,C}. QED

The following pertains to 6,2' and 6,5'.

LEMMA 3.6.25. $C \cup. fA \subseteq A \cup. gC$, $A \cap gA = \emptyset$ has ¬NON.

Proof: Let f be as given by Lemma 3.2.1. Define $g \in ELG$ by $g(n) = 2n+1$. Let $C \cup. fA \subseteq A \cup. gC$, $A \cap gA = \emptyset$, where A,B,C are nonempty.

We claim that $fA \cap 2N \subseteq A$. To see this, let $n \in fA \cap 2N$. Then $n \notin gC$, and so $n \in A$.

By Lemma 3.2.1, fA is cofinite. Hence C is finite. Therefore gC is finite. Hence A is cofinite. Therefore gA is infinite. This contradicts $A \cap gA = \emptyset$. QED