3.7. AABB.

Recall the reduced AA table from section 3.4.

REDUCED AA

1. B ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
2. B ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
3. B ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
4. C ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. C ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. C ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

The reduced BB table is obtained from the reduced AA table by interchanging A,B. We use 1'-6' to avoid any confusion. We use 1′-6′ to avoid any confusion.

REDUCED BB

1′. A ∪. fB ⊆ B ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
2′. A ∪. fB ⊆ B ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
3′. A ∪. fB ⊆ B ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.
4′. C ∪. fB ⊆ B ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
5′. C ∪. fB ⊆ B ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
6′. C ∪. fB ⊆ B ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

LEMMA 3.7.1. X ∪. fA ⊆ A ∪. gY, Z ∪. fB ⊆ B ∪. gW has ¬NON, provided X = B or Z = A.

Proof: Let f be as given by Lemma 3.2.1. Define g ∈ ELG by g(n) = 2n+1. Let X ∪. fA ⊆ A ∪. gY, Z ∪. fB ⊆ B ∪. W, where A,B,C are nonempty. Assume X = B or Z = A.

Clearly fA ∩ 2N ⊆ A and fB ∩ 2N ⊆ B. By Lemma 3.2.1, fA and fB are cofinite. Hence A,B are infinite. Since X ∩ fA = ∅, we see that X is finite. Since Z ∩ fB = ∅, we see that Z is finite. Hence A is finite or B is finite. This is a contradiction. QED

By Lemma 3.7.1, we can eliminate B ∪. fA ⊆ A ∪. gX from consideration. For the same reason, we can eliminate A ∪. fB ⊆ B ∪. gX from consideration. Thus we need only handle the two tables

4. C ∪. fA ⊆ A ∪. gA. ¬INF. AL. ¬ALF. ¬FIN. NON.
5. C ∪. fA ⊆ A ∪. gB. ¬INF. AL. ¬ALF. ¬FIN. NON.
6. C ∪. fA ⊆ A ∪. gC. ¬INF. AL. ¬ALF. ¬FIN. NON.

and

4′. $C \cup. fB \subseteq B \cup. gB$. ¬INF. AL. ¬ALF. ¬FIN. NON.
5′. $C \cup. fB \subseteq B \cup. gA$. ¬INF. AL. ¬ALF. ¬FIN. NON.
6′. $C \cup. fB \subseteq B \cup. gC$. ¬INF. AL. ¬ALF. ¬FIN. NON.

It is clear by switching A,B, that i,j′ and i′,j are equivalent, where $4 \leq i,j \leq 6$. Hence we need only consider i,j′, where $i \leq j'$.

4,4′. $C \cup. fA \subseteq A \cup. gA$, $C \cup. fB \subseteq B \cup. gB$. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,5′. $C \cup. fA \subseteq A \cup. gA$, $C \cup. fB \subseteq B \cup. gA$. ¬INF. AL. ¬ALF. ¬FIN. NON.
4,6′. $C \cup. fA \subseteq A \cup. gA$, $C \cup. fB \subseteq B \cup. gC$. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,5′. $C \cup. fA \subseteq A \cup. gB$, $C \cup. fB \subseteq B \cup. gA$. ¬INF. AL. ¬ALF. ¬FIN. NON.
5,6′. $C \cup. fA \subseteq A \cup. gB$, $C \cup. fB \subseteq B \cup. gC$. ¬INF. AL. ¬ALF. ¬FIN. NON.
6,6′. $C \cup. fA \subseteq A \cup. gC$, $C \cup. fB \subseteq B \cup. gC$. ¬INF. AL. ¬ALF. ¬FIN. NON.

As before, all proposition attributes are determined from the above tables, except for AL and NON. So we merely have to determine the status of AL and NON.

LEMMA 3.7.2. 4,4′, 4,5′, 5,5′ have AL.

Proof: From the reduced AA table, $C \cup. fA \subseteq A \cup. gA$ has AL. In the cited pairs, replace B by A. QED

The following pertains to 4,6′.

LEMMA 3.7.3. $C \cup. fA \subseteq A \cup. gA$, $C \cup. fB \subseteq B \cup. gC$ has AL.

Proof: Let $f,g \in$ ELG be given and $p > 0$. Let $C = [n,n+p]$, where n is sufficiently large. By Lemma 3.3.3, let A be unique such that $A \subseteq [n,\infty) \subseteq A \cup. gA$. Let $B = [n,\infty)\backslash gC$.

Clearly $C \cap fA = C \cap fB = C \cap gA = C \cap gC = \emptyset$. Hence $C \subseteq A,B$. Also $A \cap gA = B \cap gC = \emptyset$.

Clearly $C \cup fB \subseteq [n,\infty) = B \cup gC$. Also $C \cup fA \subseteq [n,\infty) = A \cup gA$. QED

The following pertains to 5,6'.

LEMMA 3.7.4. $C \cup. fA \subseteq A \cup. gB$, $C \cup. fB \subseteq B \cup. gC$ has AL.

Proof: Let $f,g \in ELG(N)$ and $p > 0$. Let $C = [n,n+p]$, where n is sufficiently large. Let $B = [n,\infty)\backslash gC$ and $A = [n,\infty)\backslash gB$.

Obviously $C \cap fA = C \cap fB = C \cap gC = C \cap gB = A \cap gB = B \cap gC = \emptyset$. Hence $C \subseteq A,B$. Furthermore, $fA \subseteq [n,\infty) \subseteq A \cup gB$, and $fB \subseteq [n,\infty) \subseteq B \cup gC$. QED

The following pertains to 6,6'.

LEMMA 3.7.5. $C \cup. fA \subseteq A \cup. gC$, $C \cup. fB \subseteq B \cup. gC$ has AL.

Proof: From the reduced AA table, $C \cup. fA \subseteq A \cup. gC$ has AL. Replace B by A in the cited ordered pair. QED