CHAPTER 4.
PROOF OF PRINCIPAL EXOTIC CASE

4.1. Strongly Mahlo Cardinals of Finite Order.

The large cardinal properties used in this book are the strongly Mahlo cardinals of order n, where $n \in \omega$. These are defined inductively as follows.

DEFINITION 4.1.1. The strongly 0-Mahlo cardinals are the strongly inaccessible cardinals (uncountable regular strong limit cardinals).
The strongly n+1-Mahlo cardinals are the infinite cardinals all of whose closed unbounded subsets contain a strongly n-Mahlo cardinal.

It is easy to prove by induction on n that for all $n < m < \omega$, every strongly m-Mahlo cardinal is a strongly n-Mahlo cardinal.

There is a closely related notion: n-Mahlo cardinal.

DEFINITION 4.1.2. The 0-Mahlo cardinals are the weakly inaccessible cardinals (uncountable regular limit cardinals). The n+1-Mahlo cardinals are the infinite cardinals all of whose closed unbounded subsets contain an n-Mahlo cardinal.

Again, for all $n < m < \omega$, every m-Mahlo cardinal is an n-Mahlo cardinal.

NOTE: Sometimes (strongly) n-Mahlo cardinals are called (strongly) Mahlo cardinals of order $\leq n$. Also, sometimes what we call n-Mahlo cardinals are called weakly n-Mahlo cardinals.

The well known relationship between n-Mahlo cardinals and strongly n-Mahlo cardinals is given as follows.

THEOREM 4.1.1. The following is provable in ZFC. Let $n < \omega$. A cardinal is strongly n-Mahlo if and only if it is n-Mahlo and strongly inaccessible. Under the GCH, a cardinal is strongly n-Mahlo if and only if it is n-Mahlo.

Proof: For the first claim, note that it is obvious for n = 0. Assume that every strongly inaccessible n-Mahlo cardinal is strongly n-Mahlo. Let κ be a strongly inaccessible n+1-Mahlo cardinal. Let $A \subseteq \kappa$ be closed and unbounded. Since κ is strongly inaccessible, the set $B \subseteq \kappa$ consisting of the strong limit cardinals in A is closed and unbounded. Let $\lambda \in B$ be an n-Mahlo cardinal. As previously remarked, λ is an inaccessible cardinal. Since λ is a strong limit cardinal, λ is a strongly inaccessible cardinal. By the induction hypothesis, λ is a strongly n-Mahlo cardinal.

We have thus shown that every closed unbounded $A \subseteq \kappa$ contains a strongly n-Mahlo element. Hence κ is strongly n+1-Mahlo.

For the final claim, assume the GCH. By an obvious induction, every strongly n-Mahlo cardinal is an n-Mahlo cardinal. For the converse, let κ be an n-Mahlo cardinal. As previously remarked, κ is a weakly inaccessible cardinal. Hence κ is a strongly inaccessible cardinal (by GCH). By the first claim, κ is a strongly n-Mahlo cardinal. QED

We now develop the essential combinatorics of strongly Mahlo cardinals of finite order used in this Chapter.

DEFINITION 4.1.3. Let $[A]^n$ be the set of all n element subsets of A. Sometimes we write $x \in [A]^n$ in the form $\{x_1,...,x_n\}_<$ to indicate that the xi are strictly increasing. Let A be a set of ordinals. We say that $f:[A]^n \rightarrow On$ is regressive if and only if for all $x \in [A\backslash\{0\}]^n$, $f(x) < \min(x)$.

DEFINITION 4.1.4. We say that E is min homogenous for $f:[A]^n \rightarrow On$ if and only if $E \subseteq A$ and for all $x,y \in [E]^n$, $\min(x) = \min(y) \rightarrow f(x) = f(y)$.

LEMMA 4.1.2. Let $n \geq 0$, κ a strongly n-Mahlo cardinal, $A \subseteq \kappa$ unbounded, and $f:[A]^{n+2} \rightarrow \kappa$ be regressive. For all $\alpha < \kappa$, there exists $E \subseteq A$ of order type α which is min homogenous for f.

Proof: This result originally appeared in [Sc74], in somewhat sharper form, using different notation. We present the proof in [HKS87], p. 147, using Erdös-Rado trees.

DEFINITION 4.1.5. Let A be a set of ordinals with at least two elements. An A-tree is an irreflexive transitive relation T with field A such that

i. $\alpha \; T \; \beta \rightarrow \alpha < \beta$.
ii. $\{\beta: \beta \; T \; \alpha\}$ is linearly (and hence well) ordered by T.

DEFINITION 4.1.6. Let $m \geq 2$, A be a nonempty set of ordinals, and $f:[A]^m \rightarrow On$ be regressive. The Erdos-Rado tree ERT(f) is the unique A-tree T with field A such that for all $\alpha,\beta \in A$, $\alpha \; T \; \beta$ if and only if

i. $\alpha < \beta$.
ii. For all $\gamma_1,\ldots,\gamma_{m-1} \; T \; \alpha$ with $\gamma_1 < \ldots < \gamma_{m-1}$, $f(\{\gamma_1,\ldots,\gamma_{m-1},\alpha\}) = f(\{\gamma_1,\ldots,\gamma_{m-1},\beta\})$.

To see that there is such a unique T, build ERT(f,α), $\alpha \in A$, by transfinite recursion on $\alpha \in A$. Here ERT(f,α) is ERT(f) restricted to $A \cap \alpha$. The details are left to the reader.

DEFINITION 4.1.7. For $\alpha \in A$, the height of α in ERT(f) is the order type of $\{\beta: \beta \; ERT(f) \; \alpha\}$. We say that $\alpha,\beta \in A$ are siblings in ERT(f) if and only if they are distinct, and have the same strict predecessors in ERT(f). For ordinals γ, let ERT(f)[$<\gamma$] be the restriction of ERT(f) to the elements of A (vertices) of height $< \gamma$.

We now assume that $f:[A]^{n+2} \rightarrow On$ is regressive and sup(A) is a strongly inaccessible cardinal κ. Observe that for all $\alpha \in A$, the number of siblings of α in ERT(f) is at most the number of functions from α^{n+1} into α, which is at most $2^{|\alpha|}+\omega$. Next observe that by transfinite induction on $\alpha < \kappa$, ERT(f)[$<\alpha$] has $< \kappa$ vertices. Hence for all $\alpha < \kappa$, ERT(f) has a vertex of height α. By the construction of ERT(f), every vertex has height $< \kappa$.

Now observe that if $n = 0$ then the set of strict predecessors of every element of ERT(f) is min homogeneous for f. This establishes the Lemma for the basis case $n = 0$.

Suppose that the Lemma holds for a fixed $n \geq 0$. Let κ be a strongly n+1-Mahlo cardinal, $A \subseteq \kappa$ be unbounded, $\alpha < \kappa$, and $f:[A]^{n+3} \to \kappa$ be regressive. We use the Erdös-Rado tree ERT(f).

Since κ is strongly inaccessible, $C = \{\lambda < \kappa: \lambda$ is a limit ordinal $> \alpha$ and ERT(f)[$<\lambda$] is an $A \cap \lambda$-tree and $A \cap \lambda$ is unbounded in $\lambda\}$ is a closed and unbounded subset of κ. Since κ is a strongly n+1-Mahlo cardinal, fix $\lambda < \kappa$ to be a strongly n-Mahlo cardinal $> \alpha$ such that ERT(f)[$<\lambda$] is an $A \cap \lambda$-tree and $A \cap \lambda$ is unbounded in λ.

Let v be a vertex of ERT(f) of height λ. Let B = {w: w ERT(f) v}. Then B is an unbounded subset of λ.

B naturally gives rise to a regressive function $f^*:[B]^{n+2} \to \lambda$ by taking $f^*(x) = f(x \cup \{\gamma\})$, where $\gamma \in B$, $\gamma > \max(x)$. Note that this definition is independent of the choice of γ.

By the induction hypothesis, let $E \subseteq B$ be min homogenous for f*, E of order type α. Then $E \subseteq B \subseteq A$ is min homogenous for f. QED

DEFINITION 4.1.8. For all ordinals α, let α^+ be the least infinite cardinal $> \alpha$. Let $f:[A]^n \to \kappa$. We say that f is next regressive if and only if every $f(x_1,\ldots,x_n) < \min(x_1,\ldots,x_n)^+$.

LEMMA 4.1.3. Let $n \geq 0$, κ a strongly n-Mahlo cardinal, and $A \subseteq \kappa$ be unbounded. For all $i \in \omega$, let $f_i:[A]^{n+2} \to \kappa$ be next regressive. For all $\alpha < \kappa$, there exists $E \subseteq A$ of order type α such that for all $i \in \omega$, E is min homogenous for f_i.

Proof: This is by a straightforward modification of the proof of Lemma 4.1.2. Modify the definition of the Erdös-Rado tree ERT(f) accordingly, and derive a similar upper bound on the number of siblings of a vertex in ERT(f). QED

Let $n \geq 1$ and $f:[A]^n \to \kappa$. We wish to define n+1 kinds of infinite sets $E \subseteq A$ for f.

DEFINITION 4.1.9. We say that E is of kind 0 for f if and only if f is constant on $[E]^n$, where the constant value is less than the strict sup of E.

DEFINITION 4.1.10. We say that E is of kind $1 \leq j \leq n$ for f if and only if the following holds. For all $\{x_1,...,x_n\}_<$, $\{x_1,...,x_j,y_{j+1},...,y_n\}_< \subseteq E$, $f(x_1,...,x_n) = f(x_1,...,x_j,y_{j+1},...,y_n)$ is greater than every element of E < x_j and smaller than every element of E > x_j.

For $E \subseteq On$ and $\delta < ot(E)$, we write $E[\delta]$ for the δ-th element of E.

We fix $H:On^{<\omega} \to On\backslash\{0\}$, where H is one-one and for all $x \in On^{<\omega}$, $H(x) < \max(x)^+$.

LEMMA 4.1.4. Let $n \geq 1$, κ a strongly n-Mahlo cardinal, and $A \subseteq \kappa$ unbounded. For all $i \in \omega$, let $f_i:[A]^{n+1} \to \kappa$. For all $\alpha < \kappa$, there exists $E \subseteq A$ of order type α such that the following holds. For all $i \in \omega$, there exists $0 \leq j \leq n+1$ such that E is of kind j for f_i.

Proof: Let n,κ,A,f_i,α be as given. We can assume that $\alpha > \omega$, $A \subseteq \kappa\backslash\omega$, and there is an infinite cardinal strictly between any two elements of A. We can also assume that for all $\alpha_1,...,\alpha_{n+1} < \beta$ from A, $f_i(\alpha_1,...,\alpha_{n+1}) < \beta$.

For all $i \in \omega$, define $g_{i,0}\{u,x_1,...,x_{n+1}\}_< = 1+f_i\{x_1,...,x_{n+1}\}$ if $f_i\{x_1,...,x_{n+1}\} \leq u$; 0 otherwise.

For $1 \leq j \leq n+1$, define $g_{i,j}\{u,x_{j+1},...,x_{n+2}\}_<$ as follows. Let $z_1 < ... < z_j \leq u$ be such that $f_i\{z_1,...,z_j,x_{j+1}...,x_{n+1}\} \neq f_i\{z_1,...,z_j,x_{j+2},...,x_{n+2}\}$ and $f_i\{z_1,...,z_j,x_{j+1}...,x_{n+1}\} \leq u$. Set $g_{i,j}\{u,x_{j+1},...,x_{n+2}\} = H(z_1,...,z_j,f_i\{z_1,...,z_j,x_{j+1},...,x_{n+1}\})$. If such z's do not exist, then set $g_{i,j}\{u,x_{j+1},...,x_{n+2}\} = 0$.

Note that each $g_{i,j}$ is next regressive. By Lemma 4.1.3, let $E' \subseteq A\backslash\omega$ be min homogeneous for all $g_{i,j}$, where E' has cardinality $\geq \beth_\omega(\alpha+\omega)$ = the first strong limit cardinal > $\alpha+\omega$.

We can partition the tuples from E' of length $\leq 2n+2$ in a strategic way, with 2^ω pieces, and apply the Erdös-Rado theorem to obtain $E \subseteq E'$ with order type α, with the following three properties. Write E[1],E[2],... for the first ω elements of E. Let $i \in \omega$.

1) For all $\{x_1,...,x_{n+1}\}_< \in [E]^{n+1}$, $f_i\{x_1,...,x_{n+1}\} \in E \to f_i\{x_1,...,x_{n+1}\} \in \{x_1,...,x_{n+1}\}$.

2) Suppose $f_i\{E[2],...,E[n+2]\} = f_i\{E[n+3],...,E[2n+3]\}$. Then f_i is constant on $[E]^{n+1}$.

3) Suppose $1 \leq j \leq n+1$, and $f_i\{E[2],E[4],...,E[2n+2]\} = f_i\{E[2],E[4],...,E[2j],E[2j+4],E[2j+6],...,E[2n+4]\} \in (E[2j-1],E[2j+1])$. Then E is of kind j for f_i.

For the remainder of the proof, we fix $i \in \omega$. The first case that applies is the operative case.

case 1. $f_i\{E[2],E[4],...,E[2n+2]\} \leq E[1]$. Then $g_{i,0}\{E[1],E[2],E[4],...,E[2n+2]\} = 1+f_i\{E[2],E[4],...,E[2n+2]\} > 0$. Since E is min homogenous for $g_{i,0}$ we see that for all $x,y \in [E]^{n+1}$ such that $\min(x),\min(y) \geq E[2]$, we have $g_{i,0}(\{E[1]\} \cup x) = g_{i,0}(\{E[1]\} \cup y) = 1+f_i(x) = 1+f_i(y)$. In particular, $f_i\{E[2],...,E[n+2]\} = f_i\{E[n+3],...,E[2n+3]\}$. By 2), f_i is constant on $[E]^{n+1}$. Hence E is of kind 0 for f_i.

case 2. Let j be the greatest element of [1,n+1] such that $f_i\{E[2],E[4],...,E[2n+2]\} \in (E[2j-1],E[2j+1])$. Note that $g_{i,j}\{E[2j+1],E[2j+2],E[2j+4],...,E[2n+4]\} = g_{i,j}\{E[2j+1],E[2j+4],E[2j+6],...,E[2n+6]\}$.

Suppose the main clause in the definition of $g_{i,j}\{E[2j+1],E[2j+2],E[2j+4],...,E[2n+4]\}$ holds, with $z_1 < ... < z_j \leq E[2j+1]$. Since H is nonzero, the main clause in the definition of $g_{i,j}\{E[2j+1],E[2j+4],E[2j+6],...,E[2n+6]\}$ holds with, say, $w_1 < ... < w_j \leq E[2j+1]$. Hence $H(z_1,...,z_j,f_i\{z_1,...,z_j,E[2j+2],E[2j+4],...,E[2n+2]\}) = H(w_1,...,w_j,f_i\{w_1,...,w_j,E[2j+4],E[2j+6],...,E[2n+4]\})$. Therefore $z_1,...,z_j = w_1,...,w_j$, respectively, and $f_i\{z_1,...,z_j,E[2j+2],E[2j+4],...,E[2n+2]\} = f_i\{w_1,...,w_j,E[2j+4],E[2j+6],...,E[2n+4]\}$. This contradicts the choice of $z_1,...,z_j$.

Hence the main clause in the definition of $g_{i,j}\{E[2j+1],E[2j+2],E[2j+4],...,E[2n+4]\}$ fails. In particular, it fails with $z_1,...,z_j = E[2],E[4],...,E[2j]$, respectively. Then $f_i\{E[2],E[4],...,E[2n+2]\} = f_i\{E[2],E[4],...,E[2j],E[2j+4],E[2j+6],...,E[2n+4]\}$. By 3), E is of kind j for f_i.

case 3. Otherwise. Then $f_i\{E[2],E[4],...,E[2n+2]\} \in \{E[1],E[3],...,E[2n+1]\}$, or $f_i\{E[2],E[4],...,E[2n+2]\} \geq E[2n+3]$. The first disjunct is impossible by 1), and the second disjunct is impossible by the assumption on A.

We have thus shown that for some $j \in [0,n+1]$, E is of kind j for f_i. Since i is arbitrarily chosen from ω, we are done.

QED

DEFINITION 4.1.11. Let $f:[A]^n \to \kappa$ and $E \subseteq A$. We define fE to be the range of f on $[E]^n$.

LEMMA 4.1.5. Let $n,m \geq 1$, κ a strongly n-Mahlo cardinal, and $A \subseteq \kappa$ unbounded. For all $i \in \omega$, let $f_i:[A]^{n+1} \to \kappa$, and let $g_i:[A]^m \to \omega$. There exists $E \subseteq \kappa$ of order type ω such that
i) for all $i \in \omega$, f_i is either constant on $[E]^{n+1}$, with constant value < sup(E), or f_iE is of order type ω with the same sup as E;
ii) for all $i \in \omega$, g_i is constant on $[E]^m$.

Proof: Let n,m,κ,A,f_i,g_i be as given. Apply Lemma 4.1.4 to obtain $E' \subseteq \kappa$ of order type $\Im_\omega(\omega)$ such that the following holds. For all $i \in \omega$ there exists $0 \leq j \leq n+1$ such that E is of kind j for f_i. By the Erdös-Rado theorem, let $E \subseteq E'$ be of order type ω, where for all $i \in \omega$, g_i is constant on $[E]^m$. Write $E = \{E[1],E[2],...\}_<$.

Let $i \in \omega$ and E be of kind j for f_i. If $j = 0$ then f_i is constant on $[E]^{n+1}$, where the constant value is less than sup(E).

Now suppose $1 \leq j \leq n+1$. For all $\{x_1,...,x_{n+1}\}_<$, $\{x_1,...,x_j,y_{j+1},...,y_{n+1}\}_< \subseteq E$, $f_i\{x_1,...,x_{n+1}\} = f\{x_1,...,x_j,y_{j+1},...,y_{n+1}\}$ is greater than every element of E $< x_j$ and smaller than every element of E $> x_j$. Since we can set x_j to vary among E[j],E[j+1],..., we see that f_iE has the same sup as E. In particular, f_iE is infinite.

Also, for any particular E[p], the values $f_i\{x_1,...,x_{n+1}\} < E[p]$, $x_1 < ... < x_{n+1} \in A$, can arise only if $x_j \leq E[p+1]$. Since the arguments $x_{j+1},...,x_{n+1}$ don't matter (kind j for f_i), there are at most finitely many such values.

We have shown that f_iE has at most finitely many elements not exceeding any given element of E. Therefore f_iE has order type $\leq \omega$. Since f_iE is infinite, the order type of f_iE is ω. QED

We now switch over to ordered tuples. Let $f:A^n \to \kappa$ and $E \subseteq A$. Here we also define fE to be the range of f on E^n.

LEMMA 4.1.6. Let $n,m \geq 1$, κ a strongly n-Mahlo cardinal, and $A \subseteq \kappa$ unbounded. For all $i \in \omega$, let $f_i:A^{n+1} \to \kappa$, and let $g_i:A^m \to \omega$. There exists $E \subseteq \kappa$ of order type ω such that
i) for all $i \geq 1$, f_iE is either a finite subset of sup(E), or of order type ω with the same sup as E;
ii) for all $i \in \omega$, g_iE is finite.

Proof: Let n,m,κ,A,f_i,g_i be as given. Each f_i gives rise to finitely many corresponding $f_{i,\sigma}$, where σ ranges over the order types of n+1 tuples. Also each g_i gives rise to finitely many corresponding $g_{i,\sigma}$, where σ ranges over the order types of m tuples. Any f_iE is the union of the $f_{i,\sigma}E$, and any g_iE is the union of the $g_{i,\sigma}E$. Choose E according to Lemma 4.1.5. Then E will be as required. QED

DEFINITION 4.1.12. Let $SMAH^+$ be ZFC + $(\forall n < \omega)(\exists\kappa)(\kappa$ is a strongly n-Mahlo cardinal). Let SMAH be ZFC + $\{(\exists\kappa)(\kappa$ is a strongly n-Mahlo cardinal$)\}_{n<\omega}$.

DEFINITION 4.1.13. Let MAH^+ be ZFC + $(\forall n < \omega)(\exists\kappa)(\kappa$ is an n-Mahlo cardinal). Let MAH be ZFC + $\{(\exists\kappa)(\kappa$ is an n-Mahlo cardinal$)\}_{n < \omega}$.

We will use the following (known) relationship between $SMAH^+$, MAH^+, SMAH, and MAH.

DEFINITION 4.1.14. The system EFA = exponential function arithmetic is defined to be the system $I\Sigma_0(\exp)$; see [HP93].

THEOREM 4.1.7. $SMAH^+$ and MAH^+ prove the same Π^1_2 sentences. SMAH and MAH prove the same Π^1_2 sentences. SMAH is 1-consistent if and only if MAH is 1-consistent. SMAH is consistent if and only if MAH is consistent. These results are provable in EFA.

Proof: We first prove the following well known theorem in ZFC.

1) Let $n \geq 0$. Every n-Mahlo cardinal is an n-Mahlo cardinal in the sense of L.

The basis case asserts that every weakly inaccessible cardinal is a weakly inaccessible cardinal in L. This is particularly well known and easy to check.

Fix n ≥ 0 and assume that every n-Mahlo cardinal is an n-Mahlo cardinal in L. Let κ be an n+1-Mahlo cardinal. Let A ⊆ κ, A ∈ L, where A is closed and unbounded in κ (in the sense of L). Let λ ∈ A be an n-Mahlo cardinal. Then λ ∈ A is an n-Mahlo cardinal in L. Hence κ is an n+1-Mahlo cardinal in L.

If T is a sentence or set of sentences in the language of set theory, then we write $T^{(L)}$ for the relativization of T to Godel's constructible universe L.

For the first claim, let $SMAH^+$ prove φ, where φ is Π^1_2. By Lemma 4.1.1, MAH^+ + GCH proves φ. Hence ZFC + $MAH^{+(L)}$ + $GCH^{(L)}$ proves $\varphi^{(L)}$ by, e.g., [Je78], section 12. Therefore ZFC + $MAH^{+(L)}$ proves $\varphi^{(L)}$ by, e.g., [Je78], section 13. By the Shoenfield absoluteness theorem (see, e.g., [Je78], p. 530), ZFC + $MAH^{+(L)}$ proves φ. By 1), MAH^+ proves φ.

For the second claim, we repeat the proof of the first claim for any specific level of strong Mahloness.

For the third claim, assume 1-Con(MAH). Let φ be a Σ^0_1 sentence provable in SMAH. By the second claim, φ is provable in MAH. Hence φ is true.

For the final claim, assume Con(MAH). Then MAH does not prove 1 = 0. By the second claim, SMAH does not prove 1 = 0. Hence Con(SMAH). QED

Theorem 4.1.7 tells us that for the purposes of this book, $SMAH^+$ and SMAH are equivalent to MAH^+ and MAH. We will always use $SMAH^+$ and SMAH.