4.2. Proof using Strongly Mahlo Cardinals.

Recall Proposition A from the beginning of section 3.1. This is the Principal Exotic Case.

PROPOSITION A. For all $f,g \in ELG$ there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

Recall the definitions of N, ELG, INF, ∪., fA, in Definitions 1.1.1, 1.1.2, 1.1.10, 1.3.1, and 2.1.

In this section, we prove Proposition A in $SMAH^+$. It is convenient to prove a stronger statement.

PROPOSITION B. Let $f,g \in ELG$ and $n \geq 1$. There exist infinite sets $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that
i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii) $A_1 \cap fA_n = \emptyset$.

LEMMA 4.2.1. The following is provable in RCA_0. Proposition B implies Proposition A. In fact, Proposition B for $n = 3$ implies Proposition A.

Proof: Let $f,g \in ELG$. By Proposition B for $n = 3$, let $A \subseteq B \subseteq C \subseteq N$ be infinite sets, where $fA \subseteq B \cup. gB$, $fB \subseteq C \cup. gC$, and $A \cap fC = \emptyset$.

Note that C,gC are disjoint. Hence C,gB are disjoint. In addition, A,fA are disjoint, and A,fB are disjoint. We now verify the inclusion relations.

Let $x \in A \cup fA$. If $x \in fA$ then $x \in B \cup gB \subseteq C \cup gB$. If $x \in A$ then $x \in C \subseteq C \cup gB$.

Let $x \in A \cup fB$. If $x \in fB$ then $x \in C \cup gC$. If $x \in A$ then $x \in C \subseteq C \cup gC$. QED

Recall the definition of $f \in ELG$ from section 2.1: there are rational constants $c,d > 1$ such that for all but finitely many $x \in dom(f)$, $c|x| \leq f(x) \leq d|x|$.

We wish to put this in more explicit form. Assume f,c,d are as above. Let t be a positive integer so large that $1 + 1/t < c,d < t$, and for all $x \in dom(f)$, $|x| > t \rightarrow c|x| \leq f(x) \leq$

d|x|. Let b be an integer greater than t and max{f(x): |x| ≤ t}. Then for all x ∈ dom(f),

$$|x| > t \rightarrow f(x) \leq b|x|.$$
$$|x| \leq t \rightarrow f(x) \leq b.$$
$$|x| \leq b \rightarrow f(x) \leq b^2.$$

Hence f ∈ ELG if and only if there exists a positive integer b such that for all x ∈ dom(f),

$$|x| > b \rightarrow (1 + 1/b)|x| \leq f(x) \leq b|x|.$$
$$|x| \leq b \rightarrow f(x) \leq b^2.$$

We now fix f,g ∈ ELG, where f is p-ary and g is q-ary. According to the above, we also fix a positive integer b such that for all $x \in N^p$ and $y \in N^q$,

i. if |x|,|y| > b then

$$(1 + 1/b)|x| \leq f(x) \leq b|x|$$
$$(1 + 1/b)|y| \leq g(y) \leq b|y|.$$

ii. if |x|,|y| ≤ b then $f(x),g(y) \leq b^2$.

We also fix n ≥ 1 and a strongly p^{n-1}-Mahlo cardinal κ.

We begin with the discrete linearly ordered semigroup with extra structure, M = (N,<,0,1,+,f,g).

The plan will be to first construct a structure of the form $M^* = (N^*,<^*,0^*,1^*,+^*,f^*,g^*,c_0^*,...)$, where the c*'s are indexed by N. This structure is non well founded and generated by the constants 0*,1*, and the c*'s. The indiscernibility of the c*'s will be with regard to atomic formulas only. The first nonstandard point in M* will be c_0^*.

While it is obvious that we cannot embed M* back into M, we use the fact that we can embed any partial substructure of M* that is "boundedly generated" back into M.

Of course, M* is not well founded, but we prove the well foundedness of the crucial irreflexive transitive relation

$$sx <^* y$$

on N*, where s > 1 is any fixed rational number.

Using the atomic indiscernibility of the c*'s, we canonically extend M* to a structure M** = (N**,<**,0**,1**,+**,f**,g**,c_0**,...,c_α**,...), $\alpha < \kappa$. Many properties of M* are preserved when passing to M**. The appropriate embedding property asserts that any partial substructure of M** boundedly generated by 0**,1**, and a set of c**'s of order type ω is embeddable back into M* and M.

Recall that the proof of the Complementation Theorem (Theorem 1.3.1) requires that the function is strictly dominating with respect to a well founded relation <. Here we verify that g** is strictly dominating on the nonstandard part of M** with respect to the above crucial irreflexive transitive relation. This enables us to apply the Complementation Theorem 1.3.1) to g** on the nonstandard part of M** in order to obtain a unique set W $\subseteq$ nst(M**) such that for all x $\in$ nst(M**), x $\in$ W $\leftrightarrow$ x $\notin$ g**W.

We then build a Skolem hull construction of length ω consisting entirely of elements of W. The construction starts with the set of all c**'s. Witnesses are thrown in from W that verify that values of f** at elements thrown in at previous stages do not lie in W (provided they in fact do not lie in W). Only the first n stages of the construction will be used.

Every element of the n-th stage of the Skolem hull construction has a suitable name involving e = e(p,q) of the c**'s.

At this crucial point, we then apply Lemma 4.1.6 to the large cardinal κ, with arity n = e, in order to obtain a suitably indiscernible set S of the c**'s of order type ω, with respect to this naming system.

We can redo the length n Skolem hull construction starting with S. This is just a restriction of the original Skolem hull construction that started with all of the c**'s.

Because of the indiscernibility, we generate a subset of N** whose elements are given by terms of bounded length in c**'s of order type ω. This forms a suitable partial substructure of M**, so that it is embeddable back into M. The image of this embedding on the n stages of the Skolem hull construction will comprise the $A_1 \subseteq \ldots \subseteq A_n$ satisfying the conclusion of Proposition B. This completes the description of the plan for the proof.

We now begin the detailed proof of Proposition B. We begin with the structure M = (N,<,0,1,+,f,g) in the language L consisting of the binary relation <, constants 0,1, the binary function +, the p-ary function f, the q-ary function g, and equality.

DEFINITION 4.2.1. Let $V(L) = \{v_i: i \geq 0\}$ be the set of variables of L. Let TM(L) be the set of terms of L, and AF(L) be the set of atomic formulas of L. For $t \in TM(L)$, we define lth(t) as the total number of occurrences of functions, constants, and variables, in t. For $\varphi \in AF(L)$, we also define lth(φ) as the total number of occurrences of functions, constants, and variables, in φ.

DEFINITION 4.2.2. An M-assignment is a partial function $h:V(L) \to N$. We write Val(M,t,h) for the value of the term t in M at the assignment h. This is defined if and only if h is adequate for t; i.e., h is defined at all variables in t.

DEFINITION 4.2.3. We write Sat(M,φ,h) for atomic formulas φ. This is true if and only if h is adequate for φ and M satisfies φ at the assignment h. Here h is adequate for φ if and only if h is defined at (at least) all variables in φ.

DEFINITION 4.2.4. We say that a partial function $h:V(L) \to N$ is increasing if and only if for all $i < j$, if $v_i,v_j \in dom(h)$ then $h(v_i) < h(v_j)$.

LEMMA 4.2.2. There exist infinite sets $N \supseteq E_0 \supseteq E_1 \supseteq \ldots$ indexed by N, such that for all $i \geq 0$, $\varphi \in AF(L)$, $lth(\varphi) \leq i$, and increasing partial functions $h_1,h_2:V(L) \to N$ adequate for φ with $rng(h_1),rng(h_2) \subseteq E_i$, we have $Sat(M,\varphi,h_1) \leftrightarrow Sat(M,\varphi,h_2)$.

Proof: A straightforward application of the usual infinite Ramsey theorem, repeated infinitely many times. Each E_{i+1} is obtained by Ramsey’s theorem applied to a coloring of i-tuples from E_i. QED

DEFINITION 4.2.5. We fix the E’s in Lemma 4.2.2. In an abuse of notation, we write Sat(M,φ,E) if and only if $\varphi \in$

AF(L) and for all increasing h adequate for φ with range included in E_i, we have Sat(M,φ,h), where lth(φ) = i.

Note that by Lemma 4.2.2, this is equivalent to: φ ∈ AF(L) and for some increasing h adequate for φ with range included in E_i, we have Sat(M,φ,h), where lth(φ) = i. We can also use any i with i ≥ lth(φ) and get an equivalent definition of Sat(M,φ,E).

DEFINITION 4.2.6. We now introduce constants c_i, i ∈ N. Let C be the set of all such constants. Let L* be L expanded by these constants. Structures for L* will be written M* = (N*,<*,0*,1*,+*,f*,g*,c_0*,...). Here each c_i is interpreted by c_i*.

DEFINITION 4.2.7. We let CT(L*) be the set of closed terms of L*, and AS(L*) be the set of atomic sentences of L*. We define lth(t), lth(φ) for t ∈ AS(L*), φ ∈ AS(L*).

DEFINITION 4.2.8. For φ ∈ AS(L*), t ∈ CT(L*), we write Sat(M*,φ) and Val(M*,t) for the usual model theoretic notions.

For each t ∈ CT(L*), let X(t) ∈ TM(L) be the result of replacing all occurrences of 'c' by 'v'. For each φ ∈ AS(L*), let X(φ) ∈ AF(L) be the result of replacing all occurrences of 'c' by 'v'.

DEFINITION 4.2.9. Let T = {φ ∈ AS(L*): Sat(M,X(φ),E)}.

LEMMA 4.2.3. T is consistent. For all s,t ∈ CT(L*), exactly one of s = t, s < t, t < s belongs to T. For all n ∈ N, $c_n < c_{n+1}$ ∈ T.

Proof: It suffices to show that every finite subset of T is consistent. Let $φ_1,...,φ_k$ ∈ T. Then each Sat(M,X($φ_i$),E) holds. Let j = max(lth($φ_1$),...,lth($φ_k$)) and h:V(L) → E_j be the increasing bijection. Then each Sat(M,X($φ_i$),h) holds. Let M' be the expansion of M that interprets each constant c_n as h(v_n). Then each Sat(M',$φ_i$) holds.

For the second claim, let s,t ∈ CT(L*). Let i = lth(s = t) and h:V(L) → E_i be the increasing bijection. Then Sat(M,X(s = t),h) or Sat(M,X(s < t),h) or Sat(M,X(t < s),h). Therefore at least one of s = t, s < t, t < s lies in T. Since at most one of Sat(M,X(s = t),E), Sat(M,X(s < t),E), Sat(M,X(t < s),E) can hold, clearly at most one of s = t, s < t, t < s lies in T.

For the third claim, let n ∈ N, and let $h:V(L) \to E_2$ be the increasing bijection. Obviously $Sat(M,v_n < v_{n+1},h)$. Hence $c_n < c_{n+1} \in T$. QED

We now fix $M^* = (N^*,0^*,1^*,<^*,+^*,f^*,g^*,c_0^*,\ldots)$ to be any model of T which is generated from its constants. Such an M* exists by Lemma 4.2.3 and the fact that T consists entirely of atomic sentences. Clearly M* is unique up to isomorphism.

DEFINITION 4.2.10. For d ∈ N and t ∈ CT(L*) or t ∈ TM(L). Define dt to be the term

$$t + t + \ldots + t$$

associated to the left, where there are d t's. If d = 0, then take dt to be 0. Obviously dt ∈ CT(L*) or dt ∈ TM(L), respectively.

LEMMA 4.2.4. Let φ ∈ AS(L*). Sat(M*,φ) if and only if φ ∈ T. <* is a linear ordering on N*. For all n,d ∈ N, $dc_n < c_{n+1} \in T$.

Proof: Since M* satisfies T, the reverse direction of the first claim is immediate.

Suppose φ ∉ T. First assume φ is of the form s < t. By Lemma 4.2.3, t < s ∈ T or s = t ∈ T. Then Sat(M*,t < s) or Sat(M*,s = t). Therefore Sat(M*,φ) is false. Now assume φ is of the form s = t. By Lemma 4.2.3, s < t ∈ T or t < s ∈ T. Hence Sat(M*,s < t) or Sat(M*,t < s). Therefore Sat(M*,φ) is false.

The second claim follows immediately from the first claim and the second claim of Lemma 4.2.3.

For the third claim, let $i = lth(dc_n < c_{n+1})$. The unique increasing bijection $h:V(L) \to E_i$ has $dh(v_n) < h(v_{n+1})$. Hence $Sat(M,dv_n < v_{n+1},h)$, $Sat(M,dv_n < v_{n+1},E)$, and $X(dc_n < c_{n+1}) = dv_n < v_{n+1}$. Hence $dc_n < c_{n+1} \in T$. QED

DEFINITION 4.2.11. For r ≥ 1, we write M*[r] for the set of all values in M* of the terms t ∈ CT(L*) of length ≤ r.

DEFINITION 4.2.12. We say that H is an r-embedding from M* into M if and only if

i) $H:M^*[r(p+q+1)] \rightarrow N$;
ii) $H(0^*) = 0$, $H(1^*) = 1$;
iii) for all $x,y \in M^*[r(p+q+1)]$, $x <^* y \leftrightarrow H(x) < H(y)$;
iv) for all $x,y \in M^*[r]$, $H(x+^*y) = H(x)+H(y)$.
v) for all $x_1,\ldots,x_p \in M^*[r]$, $H(f^*(x_1,\ldots,x_p)) = f(H(x_1),\ldots,H(x_p))$;
vi) for all $x_1,\ldots,x_q, \in M^*[r]$, $H(g^*(x_1,\ldots,x_q)) = g(H(x_1),\ldots,H(x_q))$.

Note that by the second claim of Lemma 4.2.4, iii) implies that H is one-one.

LEMMA 4.2.5. For all $r \geq 1$, there exists an r-embedding H from M* into M.

Proof: Let $r \geq 1$ and $h:V(L) \rightarrow E_{2r(p+q+1)}$ be the unique increasing bijection.

We define $H:M^*[r(p+q+1)] \rightarrow N$ as follows. Let $x = \text{Val}(M^*,t)$, where $t \in CT(L^*)$, $\text{lth}(t) \leq r(p+q+1)$. Define $H(x) = \text{Val}(M,X(t),h)$.

To see that H is well defined, let $x = \text{Val}(M^*,t')$, where $t' \in CT(L^*)$, $\text{lth}(t') \leq r(p+q+1)$. We must verify that $\text{Val}(M,X(t),h) = \text{Val}(M,X(t'),h)$. Since $\text{lth}(t = t') \leq 2r(p+q+1)$,

$$\begin{gathered}\text{Val}(M,X(t),h) = \text{Val}(M,X(t'),h) \leftrightarrow \\ \text{Sat}(M,X(t = t'),E) \leftrightarrow \\ t = t' \in T \leftrightarrow \\ \text{Sat}(M^*,t = t') \leftrightarrow \\ \text{Val}(M^*,t) = \text{Val}(M^*,t') \leftrightarrow \\ x = x.\end{gathered}$$

For ii), $H(0^*) = \text{Val}(M,X(0),h) = 0$. $H(1^*) = \text{Val}(M,X(1),h) = 1$. Also, $c_i^* = \text{Val}(M^*,c_i)$, $H(c_i^*) = \text{Val}(M,X(c_i),h) = \text{Val}(M,v_i,h) = h(v_i) \in E_{r(p+q+1)}$.

For iii), we must verify that for $\text{lth}(t),\text{lth}(t') \leq r(p+q+1)$, $\text{Val}(M^*,t) <^* \text{Val}(M^*,t') \leftrightarrow \text{Val}(M,X(t),h) < \text{Val}(M,X(t'),h)$. Using Lemma 4.2.4, the left side is equivalent to $\text{Sat}(M^*,t < t')$, and to $t < t' \in T$. The right side is equivalent to $\text{Sat}(M,X(t < t'),h)$, to $\text{Sat}(M,X(t < t'),E)$, and to $t < t' \in T$, using $\text{lth}(t < t') \leq 2r(p+q+1)$.

For iv), we must verify that for lth(t),lth(t') ≤ r, H(Val(M*,t)+*Val(M*,t')) = H(Val(M*,t))+H(Val(M*,t')). Since lth(t+t') ≤ 2r ≤ r(p+q+1), the left side is H(Val(M*,t+t')) = Val(M,X(t+t'),h). The right side is Val(M,X(t),h)+Val(M,X(t'),h). Equality is immediate.

For v), we must verify that for lth(t_1),...,lth(t_p) ≤ r, H(f*(Val(M*,t_1),...,Val(M*,t_p)) = f(H(Val(M*,t_1)),...,H(Val(M*,t_p))). Since lth(f(t_1,...,t_p)) ≤ r(p+q+1), the left side is H(Val(M*,f(t_1,...,t_p))) = Val(M,X(f(t_1,...,t_p)),h). The right side is f(Val(M,t_1,h),...,Val(M,t_p,h)). Equality is immediate.

For vi), see v). QED

DEFINITION 4.2.13. For quantifier free formulas φ in L*, we define lth'(φ) as the total number of occurrences of functions, constants, and variables. We do not count the occurrences of connectives for lth'.

LEMMA 4.2.6. For all r ≥ 1, there is an r-embedding from M* into M with the following properties.
i. each H(c_i*) ∈ $E_{2r(p+q+1)}$.
ii if t ∈ CT(L*), lth(t) ≤ r(p+q+1), then H(Val(M*,t)) = Val(M,X(t),h).
iii. if φ ∈ AS(L*), lth(φ) ≤ r(p+q+1), then Sat(M*,φ) ↔ Sat(M,X(φ),E).
iv. if φ is a quantifier free sentence in L*, lth'(φ) ≤ r(p+q+1), then Sat(M*,φ) ↔ Sat(M,X(φ),E).

Proof: Let H:M*[r(p+q+1)] → N be an r-embedding of M* into M, constructed in the proof of Lemma 4.2.5, using the strictly increasing bijection h:V(L) → $E_{2r(p+q+1)}$. Then each H(c_i*) ∈ $E_{2r(p+q+1)}$. Let t ∈ CT(L*), lth(t) ≤ r(p+q+1). Then H(Val(M*,t)) = Val(M,X(t),h) by definition. Let φ ∈ AS(L*), lth(φ) ≤ r(p+q+1). Then Sat(M*,s = t) ↔ Val(M*,s) = Val(M*,t) ↔ Val(M,X(s),h) = Val(M,X(t),h) ↔ Sat(M,X(s = t),E). We can use < in place of =. Finally, iv follows from iii. QED

LEMMA 4.2.7. Every universal sentence of L that holds in M holds in M*. For any quantifier free sentence of L*, if we replace equal c*'s by equal c*'s in a manner that is order preserving on indices, then the truth value in M* is preserved. The c*'s are strictly increasing and unbounded in N*.

Proof: For the first claim, let $(\forall v_1)\ldots(\forall v_m)(\varphi)$ be a universal sentence of L that holds in M. Suppose it fails in M*. Let $v_1,\ldots,v_m \in N^*$, where $\varphi(v_1,\ldots,v_m)$ fails in M*. Let $t_1,\ldots,t_m \in CT(L^*)$ be such that each $v_i = Val(M^*,t_i)$. Let $lth(\varphi(t_1,\ldots,t_m)) \leq r$.

By Lemmas 4.2.5 and 4.2.6, let $H:M^*[r] \to N$ be an r-embedding of M* into M. By the final claim of Lemma 4.2.6, since not $Sat(M^*,\varphi(t_1,\ldots,t_m))$, we have not $Sat(M,X(\varphi(t_1,\ldots,t_m)),E)$. This contradicts $Sat(M,(\forall v_1)\ldots(\forall v_m)(\varphi))$.

For the second claim, let $\varphi \in AS(L^*)$. Let ψ be obtained from φ by replacing equal c*'s by equal c*'s in an order preserving way. Let $lth(\varphi) \leq r$. By Lemmas 4.2.5 and 4.2.6, let $H:M^*[r] \to N$ be an r-embedding of M* into M. By Lemma 4.2.6,

$$Sat(M^*,\varphi) \leftrightarrow Sat(M,X(\varphi),E).$$
$$Sat(M^*,\psi) \leftrightarrow Sat(M,X(\psi),E).$$

Since $X(\psi)$ is obtained from $X(\varphi)$ by replacing equal v_i's by equal v_i's in an order preserving way, the right sides of the above two equivalences are equivalent. Hence the left sides are also equivalent.

For the third claim, let $i < j$. Let $h:\{i,j\} \to E_2$ be increasing. Since $Sat(M,X(c_i < c_j),h)$, we have $Sat(M,X(c_i < c_j),E)$, and so $c_i < c_j \in T$ and $Sat(M^*,c_i < c_j)$. Hence $c_i^* <^* c_j^*$.

To see that the c*'s are unbounded in N*, let $x \in N^*$, and let $t \in CT(L^*)$ be such that $x = Val(M^*,t)$. Let c_i be the largest element of C appearing in t. We claim that $t < c_{i+1}$ lies in T. To see this, let $r = lth(t < c_{i+1})$ and $h:V(L) \to E_r$ be strictly increasing, where $h(v_{i+1}) >^* Val(M,t,h)$. Then $Sat(M,X(t < c_{i+1}),h)$, and so $Sat(M,X(t < c_{i+1}),E)$, and hence $t < c_{i+1} \in T$. Therefore $Val(M^*,t) <^* c_{i+1}^*$. QED

DEFINITION 4.2.14. Let $C' = \{c_\alpha: \alpha < \kappa\}$. C' is the set of transfinite constants. Note that $C \subseteq C'$.

DEFINITION 4.2.15. Let L** be the language L extended by constants c_α, $\alpha < \kappa$. Note that the c_i in L* are already present in L**. The new constants are the c_α, $\omega \leq \alpha < \kappa$.

DEFINITION 4.2.16. Let CT(L**) be the set of all closed terms of L**. Let AS(L**) be the set of all atomic sentences of L**.

DEFINITION 4.2.17. A reduction is a partial function J:C′ → C, where for all $\alpha < \beta$ and $i,j < \omega$, if $J(c_\alpha) = c_i$ and $J(c_\beta) = c_j$, then $i < j$. Any reduction J extends to a partial map from CT(L**) into CT(L*), and to a partial map AS(L**) into AS(L*) in the obvious way. Here J is defined at a closed term or atomic sentence of L** if and only if J is defined at every constant appearing in that closed term or atomic sentence.

DEFINITION 4.2.18. For s,t ∈ CT(L**), we define s ≡ t if and only if for all reductions J defined at s,t, Sat(M*,J(s = t)).

LEMMA 4.2.8. Let s,t ∈ CT(L**) and J,J′ be reductions defined at s,t ∈ CT(L**). Then Sat(M*,J(s = t)) ↔ Sat(M*,J′(s = t)), and Sat(M*,J(s < t)) ↔ Sat(M*,J′(s < t)). ≡ is an equivalence relation on CT(L**).

Proof: Let s,t,J,J′ be as given. Then J(s = t) and J′(s = t) are the same up to an increasing change in the c′s appearing in s, as in the second claim of Lemma 4.2.7. Hence by the second claim of Lemma 4.2.7, Sat(M*,J(s = t)) ↔ Sat(M*,J′(s = t)), and Sat(M*,J(s < t)) ↔ Sat(M*,J′(s < t)).

For the second claim, obviously ≡ is reflexive and symmetric. Now suppose s ≡ t and t ≡ r. Let J be any increasing reduction defined at s,t,r. Then Sat(M*,J(s = t)) and Sat(M*,J(t = r)). Hence Sat(M*,J(s = r)). Therefore s ≡ r. QED

DEFINITION 4.2.19. We now define the structure M** = (N**,<**,0**,1**,+**,f**,g**,c_0**,...,c_α**,...), $\alpha < \kappa$. Here the interpretation of < is <**, of 0 is 0**, of 1 is 1**, of f is f**, of g is g**, and of each c_α is c_α**.

DEFINITION 4.2.20. We will define M** as a stretching of M*. We define N** to be the set of all equivalence classes of terms in CT(L**) under the ≡ of Lemma 4.2.8. We define 0** = [0]. We define 1** = [1]. We define c_α** = [c_α].

We define [s] <** [t] if and only if Sat(M*,J(s < t)), where J is any (some) reduction defined at s,t.

We define [s] +** [t] = [s + t].

We define $f^{**}([t_1],\ldots,[t_p]) = [f(t_1,\ldots,t_p)]$.

We define $g^{**}([t_1],\ldots,[t_q]) = [g(t_1,\ldots,t_q)]$.

DEFINITION 4.2.21. For $t \in CT(L^{**})$ and $d \in N$, we write dt for t + ... + t, where there are d t's, associated to the left. If d = 0, then use 0.

DEFINITION 4.2.22. For $x \in N^{**}$ and $d \in N$, we write dx for x +** ... +** x, where there are d x's associated to the left. If d = 0, then use 0.

LEMMA 4.2.9. These definitions of <**, +**, f**, g** are well defined. For all $\alpha < \beta < \kappa$ and $d \in N$, $dc_\alpha^{**} <^{**} c_\beta^{**}$.

Proof: Suppose s ≡ s', t ≡ t'. We freely use Lemma 4.2.8.

Suppose Sat(M*,J(s < t)) holds for all reductions J defined at s,t. Let s ≡ s' and t ≡ t'. Let J' be any reduction defined at s,s',t,t'. Then Sat(M*,J'(s < t)), Sat(M*,J'(s = s')), and Sat(M*,J'(t = t')). Hence Sat(M*,J'(s' < t')). By Lemma 4.2.8, for all reductions J'' defined at s',t', Sat(M*,J''(s < t)).

Suppose s ≡ s', t ≡ t'. We want to show s + t ≡ s' + t'. Obviously for all reductions J defined at s,t,s',t', Sat(M*,J(s + t = s' + t')).

Suppose $s_1 \equiv t_1, \ldots, s_p \equiv t_p$. We want to show $f(s_1,\ldots,s_p) \equiv f(t_1,\ldots,t_p)$. Obviously for all reductions J defined at $s_1,\ldots,s_p,t_1,\ldots,t_p$, $Val(M^*,J(f(s_1,\ldots,s_p))) = Val(M^*,J(f(t_1,\ldots,t_p)))$. Hence $f(s_1,\ldots,s_p) \equiv f(t_1,\ldots,t_p)$.

The remaining case with g is handled analogously.

For the second claim, let $\alpha < \beta < \kappa$, $d \in N$, and J be any reduction defined at $dc_\alpha < c_\beta$, where $J(c_\alpha) = c_n$ and $J(c_\beta) = c_m$, $n < m$. Then $dc_\alpha^{**} <^{**} c_\beta^{**} \leftrightarrow [dc_\alpha] <^{**} [c_\beta] \leftrightarrow Sat(M^*,J(dc_\alpha < c_\beta)) \leftrightarrow Sat(M^*,dc_n < c_m)$, which holds by Lemma 4.2.4. QED

We write $M^{**} = (N^{**},<^{**},0^{**},1^{**},+^{**},f^{**},g^{**},c_0^{**},\ldots,c_\alpha^{**},\ldots)$, $\alpha < \kappa$.

The terms t ∈ CT(L**) play a dual role. We used them to define N** as the set of all [t], t ∈ CT(L**), under the equivalence relation ≡.

However, now that we have defined the structure M**, we can use the terms t ∈ CT(L**) in the expression Val(M**,t).

LEMMA 4.2.10. For all t ∈ CT(L**), Val(M**,t) = [t]. In particular, every element of N** is generated in M** from the set of all constants of M**, which is C′ ∪ {0,1}.

Proof: By induction on lth(t). QED

DEFINITION 4.2.23. Let S ⊆ κ. The S-constants are the c_α, α ∈ S. The S-terms are the t ∈ CT(L**), where all transfinite constants in t are S-constants.

LEMMA 4.2.11. Let S ⊆ κ. {[t]: t is an S-term} contains 0**, 1**, the c_α**, α ∈ S, and is closed under +**, f**, g**.

Proof: Let S ⊆ κ. Since 0,1,c_α, α ∈ S, are S-terms, we can obviously form [0],[1],[c_α], α ∈ S, which are, respectively, 0**,1**,c_α**, α ∈ S. Now let s,t be S-terms. Then [s] +** [t] = [s + t], and s + t is an S-term. The f**,g** cases are treated in the same way. QED

By Lemma 4.2.11, we let M**<S> be the substructure of M** whose domain is {[t]: t is an S-term}, where only the interpretations of S-constants are retained. By Lemma 4.2.11, M**<S> is a structure.

DEFINITION 4.2.24. Let N**<S> = dom(M**<S>) = {[t]: t is an S-term}.

LEMMA 4.2.12. Let S ⊆ κ have order type ω. Then there is a unique isomorphism from M**<S> onto M* which maps the c_α**, α ∈ S, onto the c_n*, n ∈ N.

Proof: Let J be the unique reduction from the S-constants onto C. Define h:N**<S> → N* as follows. Let t be an S-term. Set h([t]) = Val(M*,J(t)).

To see that h is well defined, let [t] = [t′], where t,t′ are S-terms. Since J is a reduction defined at t,t′, we have Val(M*,J(t = t′)), and so Val(M*,J(t)) = Val(M*,J(t′)).

For $\alpha \in S$, $h(c_\alpha{**}) = h([c_\alpha]) = Val(M^*,J(c_\alpha)) = J(c_\alpha)^*$. This establishes that h maps the $c_\alpha{**}$, $\alpha \in S$, onto the $c_n{*}$, $n \in N$.

We now verify that h is an isomorphism from M**<S> onto M*.

Suppose h([s]) = h([t]), where s,t are S-terms. Then Val(M*,J(s)) = Val(M*,J(t)). Hence Sat(M*,J(s = t)), and so s ≡ t, [s] = [t], using Lemma 4.2.8. Hence h is one-one.

Let x ∈ N*, and write x = Val(M*,t), t ∈ CT(L*). By the construction of J, let t′ be the unique S-term such that J(t′) = t. Then h([t′]) = Val(M*,J(t′)) = Val(M*,t) = x. Hence h is onto N*.

Let s,t be S-terms. Then [s] <** [t] ↔ Val(M*,J(s)) <* Val(M*,J(t)) ↔ h([s]) <* h([t]).

$$h([s] +^{**} [t]) = h([s + t]) = Val(M^*,J(s + t)) = Val(M^*,J(s) + J(t)) = Val(M^*,J(s)) +^* Val(M^*,J(t)) = h([s]) +^* h([t]).$$

$$h(f^{**}([t_1],\ldots,[t_p])) = h([f(t_1,\ldots,t_p)]) = Val(M^*,J(f(t_1,\ldots,t_p))) = Val(M^*,f(J(t_1),\ldots,J(t_p))) = f^*(Val(M^*,J(t_1)),\ldots,Val(M^*,J(t_p))) = f^*(h([t_1]),\ldots,h([t_p])).$$

The g** case is handled analogously.

Finally,

$$h(0^{**}) = h[0] = Val(M^*,J(0)) = 0.$$
$$h(1^{**}) = h[1] = Val(M^*,J(1)) = 1.$$

The uniqueness of h follows from the fact that the 0**, 1** and $c_\alpha{**}$, $\alpha \in S$, generate N**<S> in M**<S>, and the 0*, 1* and $c_n{*}$, $n \in N$, generate N* in M*. QED

DEFINITION 4.2.25. For $S \subseteq \kappa$ and $r \geq 1$, we write $M^{**}[S,r] = \{Val(M^{**},t)$: t is an S-term of length $\leq r\}$.

DEFINITION 4.2.26. We say that H is an S,r-embedding from M** into M if and only if

i) $H:M^{**}[S,r(p+q+1)] \rightarrow N$;
ii) H(0**) = 0, H(1**) = 1;

iii) for all $x,y \in M^{**}[S,r(p+q+1)]$, $x <^{**} y \leftrightarrow H(x) < H(y)$;
iv) for all $x,y \in M^{**}[S,r]$, $H(x+^{*}y) = H(x)+H(y)$.
v) for all $x_1,\ldots,x_p \in M^{**}[S,r]$, $H(f^{**}(x_1,\ldots,x_p)) = f(H(x_1),\ldots,H(x_p))$;
vi) for all $x_1,\ldots,x_q, \in M^{**}[S,r]$, $H(g^{**}(x_1,\ldots,x_q)) = g(H(x_1),\ldots,H(x_q))$.

LEMMA 4.2.13. Let $S \subseteq \kappa$ be of order type ω and $r \geq 1$. There is an S,r-embedding from M** into M. Every universal sentence of L that holds in M holds in M**. For any atomic sentence of L**, if we replace equal transfinite constants by equal transfinite constants in a manner that is order preserving on indices, then the truth value in M** is preserved. The c_α^{**}, $\alpha \in S$, are unbounded in $M^{**}[S,r]$.

Proof: By Lemma 4.2.12, let h be the unique isomorphism h from M**<S> onto M* which maps the c_α^{**}, $\alpha \in S$, onto the c_n^*, $n \in N$. By Lemma 4.2.5, there is an r-embedding from M* into M. By composing these two mappings, we obtain the desired S,r-embedding from M** into M. The remaining claims follow from Lemma 4.2.7 by the isomorphism h. QED

We refer to the second claim of Lemma 4.2.13 as universal sentence preservation (from M to M**). We refer to the third claim of Lemma 4.2.13 as atomic indiscernibility.

DEFINITION 4.2.27. For $m \in N$, we write m^ for the term 1+...+1 with m 1's, where 0^ is 0. We say that $x \in N^{**}$ is standard if and only if it is the value in M** of some m^, $m \geq 0$. We say that $x \in N^{**}$ is nonstandard if and only if x is not standard. We write st(M**) for the standard elements of N**, and nst(M**) for the nonstandard elements of N**.

LEMMA 4.2.14. Let $x \in$ nst(M**) and $m \in N$. Then $x >^{**} m$^. $c_0^{**} \in$ nst(M**).

Proof: Let $m < \omega$. Then $(\forall x)(x \leq m \rightarrow (x = 0\hat{} \vee \ldots \vee x = m\hat{}))$ holds in M. By universal sentence preservation, it holds in M**. Let x be nonstandard in M**. Then $x \leq^{**} m$^ is impossible by the above, and hence $x >^{**} m$^.

Suppose c_0^{**} is standard, and let $c_0^{**} = m$^. By atomic indiscernibility in M**, for all $n \in N$, $c_n^{**} = m$^. This is impossible, since $\alpha < \beta \rightarrow c_\alpha^{**} < c_\beta^{**}$. QED

Obviously, (n/m)x generally makes no sense in M**, where $n,m \in N$, $m \neq 0$. We have no division operation in M**, and certainly there is no 1/2 (there is no 1/2 in M). However, we can make perfectly good sense, in M**, of equations and inequalities

$$(n/m)x = (n'/m')x$$
$$(n/m)x <^{**} (n'/m')x$$
$$(n/m)x \leq^{**} (n'/m')x$$

by interpreting them as

$$nm'x = n'mx$$
$$nm'x <^{**} n'mx$$
$$nm'x \leq^{**} n'mx.$$

Universal sentence preservation can be used to support natural reasoning in M** involving such equations and inequalities.

We have been using | | for the sup norm, or max, for elements of N^t, $t \geq 1$.

DEFINITION 4.2.28. We now use | | for elements of N** = dom(M**).

LEMMA 4.2.15. Let $x_1,\ldots,x_p,y_1,\ldots,y_q \in N^{**}$, where $|x_1,\ldots,x_p|,|y_1,\ldots,y_q| >^{**} b\hat{}$. Then

$$(1 + 1/b)|x_1,\ldots,x_p| \leq^{**} f^{**}(x_1,\ldots,x_p) \leq^{**} b|x_1,\ldots,x_p|.$$
$$(1 + 1/b)|y_1,\ldots,y_q| \leq^{**} g^{**}(y_1,\ldots,y_q) \leq^{**} b|y_1,\ldots,y_q|.$$

If $|x_1,\ldots,x_p|,|y_1,\ldots,y_q| \leq^{**} b\hat{}$, then

$$f(x_1,\ldots,x_p),g(y_1,\ldots,y_q) \leq b^2\hat{}.$$

Proof: Recall the choice of $b \in N\backslash\{0,1\}$ made at the beginning of this section. These inequalities are purely universal, and hold in M. Hence they hold in M** by universal sentence preservation. QED

DEFINITION 4.2.29. Let $t \in CT(L^{**})$. We write #(t) for the transfinite constant of greatest index that appears in t. If none appears, then we take #(t) to be -1.

LEMMA 4.2.16. Let $t \in CT(L^{**})$. #(t) = -1 ↔ Val(M**,t) is standard. There exists a positive integer d such that the following holds. Suppose #(t) = c_α. Then $c_\alpha^{**} \leq^{**} Val(M^{**},t) <^{**} dc_\alpha^{**} <^{**} c_{\alpha+1}^{**}$.

Proof: We first claim the following. Suppose $\#(t) = c_\alpha$. Then $c_\alpha{}^{**} \leq^{**} \mathrm{Val}(M^{**},t)$. This follows easily using Lemmas 4.2.14, 4.2.15, and the monotonicity of +.

Now suppose $\#(t) = -1$. Since no transfinite constants appear in t, compute $\mathrm{Val}(M,t) = m \in N$. Hence $t = m\hat{}$ holds in M. By universal sentence preservation, $t = m\hat{}$ holds in M^{**}, and so $\mathrm{Val}(M^{**},t) = m\hat{}$. Now suppose $\#(t) \neq -1$, and let $\#(t) = c_\alpha$. By the first claim in the previous paragraph, $c_\alpha{}^{**} \leq \mathrm{Val}(M^{**},t)$, and so $\mathrm{Val}(M^{**},t)$ is nonstandard.

We now prove by induction on $t \in CT(L^{**})$ that there exists $d \in N\backslash\{0\}$ such that for all $\alpha < \kappa$, if $\#(t) = c_\alpha$ then $\mathrm{Val}(M^{**},t) <^{**} dc_\alpha{}^{**}$.

This is clearly true if t is a constant of L^{**}. Let $\#(s + t) = c_\alpha$. Then $\#(s),\#(t) \leq c_\alpha$. By the induction hypothesis, let $d \in N\backslash\{0\}$ be such that $\#(s) = c_\alpha \rightarrow \mathrm{Val}(M^{**},s) <^{**} dc_\alpha{}^{**}$, and $\#(t) = c_\alpha \rightarrow \mathrm{Val}(M^{**},t) <^{**} dc_\alpha{}^{**}$. Then $\#(s + t) = c_\alpha \rightarrow \mathrm{Val}(M^{**},s + t) <^{**} 2dc_\alpha{}^{**}$.

Let $\#(f(t_1,\ldots,t_p)) = c_\alpha$. Then $\#(t_1),\ldots,\#(t_p) \leq c_\alpha$. By the induction hypothesis, let $d \in N\backslash\{0\}$ be such that for all $1 \leq i \leq p$, $\#(t_i) = c_\alpha \rightarrow \mathrm{Val}(M^{**},t_i) <^{**} dc_\alpha{}^{**}$. Let $\#(f(t1,\ldots,tp)) = c_\alpha$. By Lemma 4.2.15, $\mathrm{Val}(M^{**},f(t_1,\ldots,t_p)) <^{**} bdc_\alpha{}^{**}$. The case of $g(t_1,\ldots,t_q)$ is argued in the same way. This completes the argument by induction.

We also need to establish that for all $d \in N$ and $\alpha < \kappa$, $dc_\alpha{}^{**} <^{**} c_{\alpha+1}{}^{**}$. This is from Lemma 4.2.9. QED

LEMMA 4.2.17. $c_0{}^{**}$ is the least element of $\mathrm{nst}(M^{**})$.

Proof: By Lemma 4.2.14, $c_0{}^{**} \in \mathrm{nst}(M^{**})$. Suppose $x <^{**} c_0{}^{**}$. Write $x = \mathrm{Val}(M^{**},t)$, $t \in CT(L^{**})$. By Lemma 4.2.16, $\#(t) = -1$. By Lemma 4.2.16, x is standard. QED

LEMMA 4.2.18. Let $x_1,\ldots,x_p \in N^{**}$ and $\alpha < \kappa$. Then $f^{**}(x_1,\ldots,x_p) <^{**} c_\alpha{}^{**} \leftrightarrow x_1,\ldots,x_p <^{**} c_\alpha{}^{**}$. Let $x_1,\ldots,x_q \in N^{**}$ and $\alpha < \kappa$. Then $g^{**}(x_1,\ldots,x_q) <^{**} c_\alpha{}^{**} \leftrightarrow x_1,\ldots,x_q <^{**} c_\alpha{}^{**}$. Let $x,y \in N^{**}$ and $\alpha < \kappa$. Then $x + y <^{**} c_\alpha{}^{**} \leftrightarrow x,y < c_\alpha{}^{**}$.

Proof: Let $x_1,\ldots,x_p \in N^{**}$ and $\alpha < \kappa$. Let $t_1,\ldots,t_p \in CT(L^{**})$, where each $x_i = \mathrm{Val}(M^{**},t_i)$.

First suppose that $f^{**}(x_1,...,x_p) < c_\alpha^{**}$. By Lemma 4.2.16, $\#(f(t_1,...,t_p)) < c_\alpha$ or $\#(f(t_1,...,t_p)) = -1$. Hence for all i, $\#(t_i) < c_\alpha$ or $\#(t_i) = -1$. Fix i. Then $\#(t_i) = -1$ or for some $\beta < \alpha$, $\#(t_i) = c_\beta$. In the former case, by Lemma 4.2.16, $Val(M^{**},t_i)$ is standard, and so is $< c_\alpha^{**}$, by Lemma 4.2.17. In the latter case, $Val(M^{**},t_i) <^{**} c_{\beta+1} \leq^{**} c_\alpha^{**}$, by Lemma 4.2.16.

For the converse, assume $x_1,...,x_p <^{**} c_\alpha^{**}$. Then $Val(M^*,t_1),...,Val(M^*,t_p) <^{**} c_\alpha^{**}$. If $\alpha = 0$ then by Lemmas 4.2.16 and 4.2.17, $\#(f(t_1,...,t_p)) = -1$, and so $Val(M^{**},f(t_1,...,t_p))$ is standard. So we can assume that $\alpha > 0$. By Lemma 4.2.16, none of $\#(t_1),...,\#(t_p)$ is $\geq c_\alpha$. Hence $\#(t_1),...,\#(t_p) < c_\alpha$. Let $\beta < \alpha$, where $\#(t_1),...,\#(t_p) \leq c_\beta$. By Lemma 4.2.16, $Val(M^*,f(t_1,...,t_p)) <^{**} c_{\beta+1}^{**} \leq c_\alpha^{**}$.

The remaining two claims are established analogously. QED

DEFINITION 4.2.30. Let s be a rational number. We write $<_s^{**}$ for the relation on N^{**} given by $x <_s^{**} y \leftrightarrow sx <^{**} y$.

LEMMA 4.2.19. Let s be a rational number > 1. There exists $k \geq 1$ such that for all $x_1 <_s^{**} x_2 <_s^{**} \ldots <_s^{**} x_k$, we have $2x_1 <^{**} x_k$.

Proof: Fix s as given, and let $k \geq 1$. Using universal sentence preservation, we see that for all $x_1,...,x_k \in N^{**}$, if $x_1 <_s^{**} x_2 <_s^{**} \ldots <_s^{**} x_k$ then $x_1 <_{s'}^{**} x_k$, where s' is s^{k-1}. Choose k large enough so that $s^{k-1} \geq 2$. QED

LEMMA 4.2.20. Let s be a rational number > 1. The relation $<_s^{**}$ on N^{**} is transitive, irreflexive, and well founded.

Proof: Transitivity and irreflexivity follow from universal sentence preservation. By well foundedness, we mean that every nonempty subset of N^{**} has a $<_s^{**}$ minimal element. This is equivalent to: there is no infinite $x_1 >_s^{**} >_s^{**} x_2 >_s^{**} x_3 \ldots$.

By Lemma 4.2.19, if $<_2^{**}$ is well founded then $<_s^{**}$ is well founded. We now show that $<_2^{**}$ is well founded.

Let Y be a nonempty subset of N^{**}. Choose $t \in CT(L^{**})$ such that $\#(t)$ is least with $Val(M^{**},t) \in Y$. If $\#(t) = -1$ then Y has a standard element. Let x be the least standard element of Y. Then x is a $<_2^{**}$ minimal element of S. Therefore, we can assume without loss of generality that Y has no standard elements, and #(t) $\geq$ 0.

Let #(t) = c_α and assume Y has no $<_2$** minimal element. By Lemma 4.2.16, fix d $\in$ N\{0} such that Val(M**,t) <** dc_α**. Let t = $t_1,\ldots,t_{d+1}$ $\in$ CT(L**) be such that Val(M**,t_1) $>_2$** ... $>_2$** Val(M**,t_{d+1}), where Val(M**,t_1),...,Val(M**,t_{d+1}) $\in$ Y. Then dVal(M**,t_{d+1}) <** Val(M**,t) <** dc_α**, and so Val(M**,t_{d+1}) <** c_α**. Since Y has no standard elements, α > 0. By Lemma 4.2.16, #(t_{d+1}) < c_α, which contradicts the choice of t, α. QED

DEFINITION 4.2.31. It is convenient to set s = 1 + 1/2b for using Lemma 4.2.20.

We now apply the well foundedness of $<_s$** in an essential way.

LEMMA 4.2.21. There is a unique set W such that W = {x $\in$ nst(M**): x $\notin$ g**W}. For all $\alpha < \kappa$, c_α** $\notin$ rng(f**),rng(g**). In particular, each c_α** $\in$ W.

Proof: By Lemma 4.2.15,

$$g^{**}(x_1,\ldots,x_q) \geq_{1+(1/b)}{}^{**} |x_1,\ldots,x_q|$$
$$g^{**}(x_1,\ldots,x_q) >_s{}^{**} |x_1,\ldots,x_q|$$

holds for all $x_1,\ldots,x_q$ $\in$ nst(M**). Hence g** is strictly dominating on nst(M**). By Lemma 4.2.20, $<_s$** is well founded on nst(M**). Hence we can apply the Complementation Theorem (for well founded relations), Theorem 1.3.1. Let W be the unique set such that W = {x $\in$ nst(M**): x $\notin$ g**W}.

For the second claim, write c_α** = f**($x_1,\ldots,x_p$). By Lemma 4.2.15, each x_i <** c_α**. By Lemma 4.2.18, f**($x_1,\ldots,x_p$) <** c_α**. This is a contradiction. The same argument applies to g**.

The third claim follows immediately from the second claim. QED

We fix the unique W from Lemma 4.2.21. We will use q choice functions $F_1,\ldots,F_q$:N** $\rightarrow$ W such that for all x $\in$ g**W,

$$x = g^{**}(F_1(x),\ldots,F_q(x))$$

and for all x $\notin$ g**W,

$$F_1(x) = ... = F_q(x) = c_0**.$$

We now come to the Skolem hull construction.

DEFINITION 4.2.32. Let $E \subseteq \kappa$. Define $E[1] = \{c_\alpha**: \alpha \in E\}$. Suppose $E[1] \subseteq ... \subseteq E[k] \subseteq \kappa$ have been defined, $k \geq 1$. Define $E[k+1] = E[k] \cup (W \cap f**E[k]) \cup F_1f**E[k] \cup ... \cup F_qf**E[k]$.

LEMMA 4.2.22. Let $E \subseteq \kappa$ and $i \geq 1$. $E[i] \subseteq E[i+1] \subseteq W$. $f**E[i] \subseteq E[i+1] \cup. g**E[i+1]$. $E[1] \cap f**E[i] = \emptyset$.

Proof: Let $E \subseteq \kappa$ and $i \geq 1$. $E[i] \subseteq E[i+1] \subseteq W$ is obvious by construction and the third claim of Lemma 4.2.21. Let $x \in f**E[i]$. Since $E[i] \subseteq nst(M**)$, by Lemma 4.2.15, we have $x \in nst(M**)$.

case 1. $x \in W$. Then $x \in E[i+1]$.

case 2. $x \notin W$. Since $x \in nst(M**)$, we have $x \in g**W$. Hence $x = g**(F_1(x),...,F_q(x))$. Now each $F_i(x) \in E[i+1]$ since $x \in f**E[i]$. Hence $x \in g**E[i+1]$.

We have thus established that $f**E[i] \subseteq E[i+1] \cup g**E[i+1]$.

$E[i+1] \cap g**E[i+1] = \emptyset$ follows from $W \cap g**W = \emptyset$.

$E[1] \cap f**E[i] = \emptyset$ follows from the second claim of Lemma 4.2.21. QED

Note that Proposition B is essentially the same as Lemma 4.2.22, for $1 \leq i < n$. However Proposition B lives in N and Lemma 4.2.22 lives way up in M**. The remainder of the proof of Proposition B surrounds the choice of a suitable E such that E[n] can be suitably embedded back into M.

Recall the positive integer $e = p^{n-1}$ fixed at the beginning of this section, where κ is strongly e-Mahlo. Recall that we have also fixed $n \geq 1$.

LEMMA 4.2.23. There is an integer m depending only on p,n, such that the following holds. There exist finitely many functions $G_1,G_2,...,G_m:\kappa^e \to W$, such that for all $E \subseteq \kappa$, $E[n] = G_1E \cup ... \cup G_mE$.

Proof: We show by induction on $1 \leq i \leq n$ that there exist finitely many functions $G_1,G_2,\ldots,G_m$, where each G_i is a multivariate function from κ into W of various arities $\leq p^{i-1}$, with the desired property.

For i = 1, take $G_1:\kappa \to W$, where $G_1(\alpha) = c_\alpha^{**}$.

Suppose $G_1,\ldots,G_m$ works for fixed $1 \leq i < n$, with arities $\leq p^{i-1}$. For i+1, we start with $G_1,\ldots,G_m$ in order to generate E[i] from E. In order to generate $W \cap f^{**}E[i]$, we need finitely many functions, each built from f** composed with p of the $G_1,\ldots,G_m$. The element $c_0^{**} \in W$ is used to make sure that only values in W are generated. Each of these finitely many functions have arity at most $p(p^{i-1}) = p^i$. Each of $F_j f^{**}[E_i]$, $1 \leq j \leq q$, are generated similarly.

So arities $\leq p^{n-1}$ are sufficient for the case i = n. We can obviously arrange for all of these functions to have arity $e = p^{n-1}$ by adding dummy variables. QED

We fix the functions $G_1,\ldots,G_m$ given by Lemma 4.2.23.

We now define "term decomposition" functions $H_i:W \to \kappa$, indexed by the natural numbers. Let $x \in W$.

DEFINITION 4.2.33. To define the $H_i(x)$, first choose $t \in CT(L^{**})$ such that $Val(M^{**},t) = x$. Let $c_{\alpha\_1},c_{\alpha\_2},\ldots,c_{\alpha\_s}$ be a listing of all transfinite constants appearing in t from left to right, with repetitions allowed.

DEFINITION 4.2.34. For $x \in W$, set $H_0(x) = lth(t)$. For $1 \leq i \leq s$, set $H_i(x) = \alpha_i$. For $i > s$, set $H_i(x) = 0$.

DEFINITION 4.2.35. Finally, define functions $J_{i,j}:\kappa^e \to \kappa$, $i \geq 0$, $1 \leq j \leq m$, by $J_{i,j}(\alpha_1,\ldots,\alpha_e) = H_i(G_j(\alpha_1,\ldots,\alpha_e))$.

LEMMA 4.2.24. Let $E \subseteq \kappa$. Every element of E[n] is of the form $Val(M^{**},t)$, where the length of $t \in CT(L^{**})$ lies in $\cup\{J_{0,j}E: 1 \leq j \leq m\}$ and the transfinite constants of t have subscripts lying in $\cup\{J_{i,j}E: 1 \leq i \leq lth(t) \wedge 1 \leq j \leq m\}$.

Proof: Let $E \subseteq \kappa$ and $x \in E[n]$. By Lemma 4.2.23, let $x \in G_jE$, $1 \leq j \leq m$. Let $t \in CT(L^{**})$ be the term used to write $x = Val(M^{**},t)$ in the definition of the $H_i(x)$. Write $x = G_j(\alpha_1,\ldots,\alpha_e)$, $\alpha_1,\ldots,\alpha_e \in E$. Then $J_{0,j}(\alpha_1,\ldots,\alpha_e) = H_0(x) = lth(t)$, and $J_{1,j}(\alpha_1,\ldots,\alpha_e),J_{2,,j}(\alpha_1,\ldots,\alpha_e),\ldots,$

$J_{lth(t),j}(\alpha_1,...,\alpha_e)$ enumerates at least the subscripts of transfinite constants of t. QED

LEMMA 4.2.25. There exists $E \subseteq S \subseteq \kappa$, E,S of order type ω, and a positive integer r, such that $E[n] \subseteq M^{**}[S,r]$.

Proof: We apply Lemma 4.1.6 to the following two sequences of functions. The first is the $J_{i,j}:\kappa^e \to \kappa$, where $i \geq 1$ and $1 \leq j \leq m$ (here m is as given by Lemma 4.2.23, and depends only on p,k). The first can be construed as an infinite sequence of functions from κ^e into κ, and the second can also be construed as an infinite sequence of functions from κ into ω by infinite repetition.

By Lemma 4.1.6, let $E \subseteq \kappa$ be of order type ω such that for all $i \geq 1$ and $1 \leq j \leq m$, $J_{i,j}E$ is either a finite subset of sup(E), or has order type ω with the same sup as E, and $J_{0,j}E$ is finite.

Let $r = \max(J_{0,1}E \cup ... \cup J_{0,m}E)$. By Lemma 4.2.24, every element of E[n] is the value in M** of a closed term t of length at most r, whose transfinite constants have subscripts lying in $S = \cup\{J_{ij}E: 1 \leq i \leq lth(t) \wedge 1 \leq j \leq m\}$. I.e., $E[n] \subseteq M^{**}[S,r]$. Note that S is a finite union of sets of ordinals, each of which is either a finite subset of sup(E), or is of order type ω with the same sup as E. Since $E \subseteq S$, we see that S is of order type ω. QED

DEFINITION 4.2.36. We fix E,S,r as given by Lemma 4.2.25.

THEOREM 4.2.26. Proposition B is provable in $SMAH^+$. In fact, it is provable in MAH^+.

Proof: By Lemma 4.2.22, for all $1 \leq i < n$, $f^{**}E[i] \subseteq E[i+1] \cup. g^{**}E[i+1]$, and $E[1] \cap f^{**}E[n] = \emptyset$. By Lemma 4.2.13, there is an S,r-embedding T from M** into M. Note that $f^{**}[E[n]] \cup g^{**}[E[n]] \subseteq M^{**}[S,r(p+q)] = dom(T)$.

For $1 \leq i \leq n$, let $A_i = TE[i]$. Since $E[1] \subseteq ... \subseteq E[n]$, we have $A_1 \subseteq ... \subseteq A_n \subseteq N$. By Lemma 4.2.25, $E[n] \subseteq M^{**}[S,r]$.

We first claim that for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup gA_{i+1}$.

Let $1 \leq i < n$, and $x \in fA_i$. Write $x = f(Ty_1,...,Ty_p)$, $y_1,...,y_p \in E[i]$. Hence $Tf^{**}(y_1,...,y_p) = f(Ty_1,...,Ty_p) = x$.

By Lemma 4.2.22, $f^{**}(y_1,...,y_p) \in E[i+1] \cup g^{**}E[i+1]$. First suppose $f^{**}(y_1,...,y_p) \in E[i+1]$. Then $Tf^{**}(y_1,...,y_p) = x \in A_{i+1}$.

Secondly suppose $f^{**}(y_1,...,y_p) \in g^{**}E[i+1]$, and write $f^{**}(y_1,...,y_p) = g^{**}(z_1,...,z_q)$, where $z_1,...,z_q \in E[i+1]$. Then $Tf^{**}(y_1,...,y_p) = Tg^{**}(z_1,...,z_q) = g(Tz_1,...,Tz_q) = f(Ty_1,...,Ty_p) = x$. Hence $x \in gA_{i+1}$.

We next claim that for all $1 \leq i < n$, $A_{i+1} \cap gA_{i+1} = \emptyset$. We must verify that $TE[i+1] \cap gTE[i+1] = \emptyset$. Let $x,y_1,...,y_q \in E[i+1]$, $T(x) = g(Ty_1,...,Ty_q)$. Clearly $T(x) = Tg^{**}(y_1,...,y_q)$, and so $x = g^{**}(y_1,...,y_q)$. This contradicts $E[i+1] \cap g^{**}E[i+1] = \emptyset$.

We finally claim that $A_1 \cap fA_n = \emptyset$. Let $x \in A_1$, $y_1,...,y_p \in A_n$, $x = f(y_1,...,y_p)$. Let $x' \in E[1]$, $y_1',...,y_p' \in E[n]$, where $x = T(x')$, and $y_1,...,y_p = T(y_1'),...,T(y_p')$ respectively. Note that $Tf^{**}(y_1',...,y_p') = f(T(y_1'),...,T(y_p')) = f(y_1,...,y_p) = x = T(x')$. Therefore $x' = f^{**}(y_1',...,y_p')$, contradicting the last claim of Lemma 4.2.22.

The second claim in the Lemma follows from the first by Theorem 4.1.7. This is because Proposition B is obviously in Π^1_2 form. QED

Obviously the proof of Theorem 4.2.26 gives an upper bound on the order of strongly Mahlo cardinal sufficient to prove Proposition B that depends exponentially on the arity of f and the length of the tower. Without attempting to optimize the level, we have shown the following.

COROLLARY 4.2.27. The following is provable in ZFC. Let $p,n \geq 1$. If there exists a strongly p^{n-1}-Mahlo cardinal then Proposition B holds for p-ary f, multivariate g, and n. If there exists a strongly p^2-Mahlo cardinal, then Proposition A holds for p-ary f and multivariate g. Furthermore, we can drop "strongly" from both results.

Corollary 4.2.27 is far from optimal. For instance, if n = 2 then Proposition B is provable in RCA_0, as we shall see now.

THEOREM 4.2.28. The following is provable in RCA_0. For all $f,g \in ELG$ there exist infinite $A \subseteq B \subseteq N$ such that

$$fA \subseteq B \mathbin{\dot\cup} gB$$
$$A \cap fB = \emptyset.$$

Proof: Let f,g ∈ EVSD. Let n be sufficiently large. By Theorem 3.2.5, let $A \subseteq [n,\infty)$ be infinite where $A \cap g(A \cup fA) = \emptyset$. By Lemma 3.3.3, let B be unique such that $B \subseteq A \cup fA \subseteq B \mathbin{\dot\cup} gB$. Then $A \cap gB \subseteq A \cap g(A \cup fA) = \emptyset$, and hence $A \subseteq B$. Also $A \cap fB \subseteq A \cap f(A \cup fA) = \emptyset$, and $fA \subseteq B \mathbin{\dot\cup} gB$. QED