CHAPTER 5
INDEPENDENCE OF EXOTIC CASE

5.1. Proposition C and length 3 towers.

In sections 5.1 - 5.9 we show that Proposition A implies the 1-consistency of SMAH (ZFC with strongly Mahlo cardinals of every specific finite order). The derivation is obviously conducted in ZFC. With some detailed examination, we see that this derivation can be carried out in the system ACA' used in Chapter 4. For a detailed discussion of RCA_0 and other subsystems of second order arithmetic, see [Si99].

We actually show that the specialization of Proposition A to rather concrete functions implies the 1-consistency of SMAH.

We use the following very basic functions on the set of all nonnegative integers N.

DEFINITION 5.1.1. We define $+,-,\bullet,\uparrow$,log as follows.
1. Addition. x+y is the usual addition.
2. Subtraction. Since we are in N, x-y is defined by the usual x-y if $x \geq y$; 0 otherwise.
3. Multiplication. $x \bullet y$ is the usual multiplication.
4. Base 2 exponentiation. $x\uparrow$ is the usual base 2 exponentiation.
5. Base 2 logarithm. Since we are in N, log(x) is the floor of the usual base 2 logarithm, with log(0) = 0.

DEFINITION 5.1.2. TM(0,1,+,−,•,↑,log) is the set of all terms built up from 0,1,+,−,•,↑,log, and variables $v_1,v_2,\ldots$.

DEFINITION 5.1.3. Each t ∈ TM(0,1,+,−,•,↑,log) gives rise to infinitely many functions, one of each arity that is at least as large as all subscripts of variables appearing in t, as follows. Let the variables of t be among $v_1,\ldots,v_k$, $k \geq 1$. Then we associate the function $f:N^k \to N$ given by

$$f(v_1,\ldots,v_k) = t(v_1,\ldots,v_k)$$

where t is interpreted according to Definition 5.1.1.

DEFINITION 5.1.4. BAF (basic functions) is the set of all functions given by terms in 0,1,+,−,•,↑,log, according to Definition 5.1.3.

It is very convenient to extend TM(0,1,+,−,•,↑,log) with definition by cases, to get an alternative description of BAF.

DEFINITION 5.1.5. ETM(0,1,+,−,•,↑,log) is the set of "extended terms" of the following form:

$$\begin{gathered} t_1 \text{ if } \varphi_1; \\ t_2 \text{ if } \varphi_2 \wedge \neg\varphi_1; \\ \ldots \\ t_n \text{ if } \varphi_n \wedge \neg\varphi_1 \wedge \ldots \wedge \neg\varphi_{n-1}; \\ t_{n+1} \text{ if } \neg\varphi_1 \wedge \ldots \wedge \neg\varphi_n. \end{gathered}$$

where $n \geq 1$, each t_i ∈ TM(0,1,+,−,•,↑,log), and each φ_i is a propositional combination of atomic formulas of the forms $s < t$, $s = t$, where s,t ∈ TM(0,1,+,−,•,↑,log).

DEFINITION 5.1.6. As in Definition 5.1.3, each t ∈ ETM(0,1,+,−,•,↑,log) gives rise to infinitely many functions, one of each arity at least as large as all subscripts of variables appearing in t.

DEFINITION 5.1.7. EBAF (extended basic functions) is the set of all functions arising in this manner from ETM(0,1,+,−,•,↑,log).

We now show that EBAF = BAF.

DEFINITION 5.1.8. We use L for the language in first order predicate calculus with equality based on the nonlogical symbols $<,0,1,+,-,\bullet,\uparrow,\log$.

Thus TM($0,1,+,-,\bullet,\uparrow,\log$) is the set of all terms in L. Also the formulas φ_i used in the extended terms above are exactly the quantifier free formulas in L.

LEMMA 5.1.1. BAF $\subseteq$ EBAF.

Proof: Let $t \in$ TM($0,1,+,-,\bullet,\uparrow,\log$), whose variables are among $v_1,\ldots,v_k$, $k \geq 1$. The function $f(v_1,\ldots,v_k) = t(v_1,\ldots,v_k)$ is also defined by

$$t \text{ if } v_1 = v_1;$$
$$t \text{ if } \neg v_1 = v_1.$$

which places f in EBAF. QED

LEMMA 5.1.2. The following functions lie in BAF.
i. $\text{neg}(x) = 1$ if $x = 0$; 0 otherwise.
ii. $\alpha(x) = 1$ if $x \geq 1$; 0 otherwise.
iii. $\text{conj}(x,y) = 1$ if $x \geq 1 \wedge y \geq 1$; 0 otherwise.
iv. $\text{disj}(x,y) = 1$ if $x \geq 1 \vee y \geq 1$; 0 otherwise.
v. $\text{les}(x,y) = 1$ if $x < y$; 0 otherwise.
vi. $\text{eq}(x,y) = 1$ if $x = y$; 0 otherwise.

Proof: Note that

$$\text{neg}(x) = 1-x.$$
$$\alpha(x) = 1-(1-x).$$
$$\text{conj}(x,y) = \alpha(x)\bullet\alpha(y).$$
$$\text{disj}(x,y) = \text{neg}(\text{conj}(\text{neg}(x),\text{neg}(y))).$$
$$\text{les}(x,y) = \alpha(y-x).$$
$$\text{eq}(x,y) = 1-((x-y)+(y-x)).$$

QED

LEMMA 5.1.3. Let φ be a quantifier free formula in L whose variables are among $v_1,\ldots,v_k$, $k \geq 1$. Then the function $f_\varphi(x_1,\ldots,x_k) = 1$ if $\varphi(x_1,\ldots,x_k)$; 0 otherwise, lies in BAF.

Proof: Fix $k \geq 1$. We can assume that φ uses only the connectives $\neg,\wedge$. We prove this by induction on φ obeying the hypotheses.

case 1. φ is $s = t$. Then $f_\varphi(v_1,\ldots,v_k) = eq(s(v_1,\ldots,v_k),t(v_1,\ldots,v_k))$.
case 2. φ is $s < t$. Then $f_\varphi(v_1,\ldots,v_k) = les(s(v_1,\ldots,v_k),t(v_1,\ldots,v_k))$.
case 3. φ is $\neg\psi$. Then $f_\varphi(v_1,\ldots,v_k) = neg(f_\psi(v_1,\ldots,v_k))$.
case 4. φ is $\psi \wedge \rho$. Then $f_\varphi(v_1,\ldots,v_k) = conj(f_\psi(v_1,\ldots,v_k),f_\rho(v_1,\ldots,v_k))$.

By Lemmas 5.1.1, 5.1.2, and the induction hypothesis, in each case the function constructed lies in BAF. QED

THEOREM 5.1.4. EBAF = BAF.

Proof: By Lemma 5.1.1, it suffices to prove EBAF $\subseteq$ BAF. Now let $f:N^k \rightarrow N$ be the function in EBAF given by $f(v_1,\ldots,v_k) =$

$$\begin{array}{c} t_1 \text{ if } \varphi_1; \\ t_2 \text{ if } \varphi_2 \wedge \neg\varphi_1; \\ \ldots \\ t_n \text{ if } \varphi_n \wedge \neg\varphi_1 \wedge \ldots \wedge \neg\varphi_{n-1}; \\ t_{n+1} \text{ if } \neg\varphi_1 \wedge \ldots \wedge \neg\varphi_n. \end{array}$$

where the variables in $t_1,\ldots,t_{n+1},\varphi_1,\ldots,\varphi_{n+1}$ are among $x_1,\ldots,x_k$, $k \geq 1$.

Then $f:N^k \rightarrow N$ is given by $f(v_1,\ldots,v_k) =$

$$f_{\varphi\_1}\bullet t_1 + \ldots + f_{\varphi\_n\wedge\neg\varphi\_1\wedge\ldots\wedge\neg\varphi\_n-1}\bullet t_n + f_{\neg\varphi\_1\wedge\ldots\wedge\neg\varphi\_n}\bullet t_{n+1}$$

using the notation of Lemma 5.1.3, with + associated to the left. Hence $f \in$ BAF by Lemma 5.1.3. QED

It is useful to know that certain functions lie in BAF. The powers of 2 are taken to be the integers 1,2,4,... .

THEOREM 5.1.5. The following functions lie in BAF.
i. All constant functions of every arity.
ii. n^x, where n is a given power of 2.
iii. The greatest power of 2 that is $\leq x$ if $x > 0$; 0 otherwise.

Proof: i. This is obvious using the term 1+...+1.
ii. Let $n = 2^k$, $k \geq 0$. Write $n^x = 2^{kx} = (kx)\uparrow = (x+\ldots+x)\uparrow$.
iii. $\log(x)\uparrow$ is the greatest power of 2 that is $\leq x$ if $x > 0$; 1 otherwise. To fix this, take $\log(x)\uparrow\dot{-}(1\dot{-}x)$.

QED

In this Chapter, we will show that the following specialization of Proposition A to these rather concrete functions implies the consistency of SMAH. Specifically,

PROPOSITION C. For all $f,g \in ELG \cap SD \cap BAF$, there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

We have carefully chosen BAF so that we can choose A,B,C to be (primitive) recursive sets. Accordingly, Proposition C becomes an explicitly Π^0_3 sentence. See Theorem 6.2.20.

We use $ELG \cap SD \cap BAF$ instead of $ELG \cap BAF$ because expansive linear growth is an asymptotic condition, and so $ELG \cap BAF$ is not included in SD. In BRT, the best course is to include both asymptotic and non asymptotic classes, as they behave differently. E.g., $A \cup. fA = U$ is correct in EBRT in A,fA on SD, but incorrect in EBRT in A,fA on ELG. The function $f(x) = 2n$, which lies in $ELG\backslash SD$, is a counterexample.

In the remainder of this chapter, we will assume Proposition C. Our aim is to construct a model of the system

$$SMAH = ZFC + \{\text{there exists a strongly k-Mahlo cardinal}\}_k.$$

Our construction will take place well within ZFC. (In section 5.9, we will analyze just what axioms are used for this entire development.) This will establish that none of Propositions A,B,C are provable in SMAH, provided SMAH is consistent. For otherwise, SMAH would prove its own consistency, and hence would be inconsistent by Gödel’s second incompleteness theorem.

DEFINITION 5.1.9. The $\Pi^0_1(L)$ sentences are the sentences in L which begin with zero or more universal quantifiers, followed by a formula ψ in which all quantifiers are bounded. I.e., all quantifiers in ψ appear, in abbreviated form, as

$$(\forall x < t)$$
$$(\exists x < t)$$

where x is a variable, t is a term in which x does not appear, and where the intended range of all variables is N.

DEFINITION 5.1.10. We use $TR(\Pi^0_1,L)$ for the set of all $\Pi^0_1(L)$ sentences that are true in N, using the interpretation in Definition 5.1.1.

We will actually establish a stronger result. Using Proposition C, we will construct a model of the system

$$SMAH + TR(\Pi^0_1,L).$$

Strictly speaking, Π^0_1 sentences are obviously not in the language of set theory. However, in weak fragments of set theory, there is the standard version of N,<,0,1,+,•,↑,log, where N is the set theoretic ω, 0 is ∅, 1 is {∅}, and <,+,-,•,↑,log are treated as sets of 2-tuples, 3-tuples, 3-tuples, 3-tuples, 2-tuples, and 2-tuples, respectively.

Accordingly, we view the system SMAH + $TR(\Pi^0_1,L)$ as a set theory that extends the system SMAH. The axioms of SMAH + $TR(\Pi^0_1,L)$ do not form a recursive set. However, this will not cause any difficulties.

DEFINITION 5.1.11. For $x \in N^r$, |x| denotes the maximum term of x.

DEFINITION 5.1.12. For E ⊆ N, we write E* for E\{min(E)}. If E = ∅ then we take E* = ∅.

The reader should not confuse our E* with the set of all finite sequences from E.

Recall Definition 1.1.3.

DEFINITION 5.1.13. For S ⊆ N and p,q ∈ N, we define

$$pS+q = \{pn+q: n \in S\}.$$

LEMMA 5.1.6. Let f,g ∈ ELG ∩ SD ∩ BAF. There exist f′,g′ ∈ ELG ∩ SD ∩ BAF such that the following holds. Let S ⊆ N.
i) g′S = g(S*) ∪ 6S+2;
ii) f'S = f(S*) ∪ g′S ∪ 6f(S*)+2 ∪ 2S*+1 ∪ 3S*+1.

Proof: Let f,g ∈ ELG ∩ SD ∩ BAF, where $f:N^p \to N$ and $g:N^q \to N$. We define $g':N^{q+1} \to N$ as follows. Let $x_1,\ldots,x_q,y \in N$.

case 1. $x_1,\ldots,x_q > y$. Set $g'(x_1,\ldots,x_q,y) = g(x_1,\ldots,x_q)$.

case 2. Otherwise. Set $g'(x_1,\ldots,x_q,y) = 6|x_1,\ldots,x_q,y|+2$.

We define $f':N^{5p+q+1} \to N$ as follows. Let $x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1} \in N$.

case a. $|y_1,\ldots,y_{q+1}| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{2p+1},\ldots,x_{3p}| = |x_{3p+1},\ldots,x_{4p}| = |x_{4p+1},\ldots,x_{5p}|$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}) = g'(y_1,\ldots,y_{q+1})$.

case b. $|y_1,\ldots,y_{q+1}| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{2p+1},\ldots,x_{3p}| = |x_{3p+1},\ldots,x_{4p}| < \min(x_{4p+1},\ldots,x_{5p})$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}) = f(x_{4p+1},\ldots,x_{5p})$.

case c. $|y_1,\ldots,y_{q+1}| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{2p+1},\ldots,x_{3p}| = |x_{4p+1},\ldots,x_{5p}| < \min(x_{3p+1},\ldots,x_{4p})$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}) = 6f(x_{3p+1},\ldots,x_{4p})+2$.

case d. $|y_1,\ldots,y_{q+1}| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{3p+1},\ldots,x_{4p}| = |x_{4p+1},\ldots,x_{5p}| < \min(x_{2p+1},\ldots,x_{3p})$. Set $f'(x_1,\ldots,x_{35},y_1,\ldots,y_{q+1}) = 2|x_{2p+1},\ldots,x_{3p}|+1$.

case e. $|y_1,\ldots,y_{q+1}| = |x_1,\ldots,x_p| = |x_{2p+1},\ldots,x_{3p}| = |x_{3p+1},\ldots,x_{4p}| = |x_{4p+1},\ldots,x_{5p}| < \min(x_{p+1},\ldots,x_{2p})$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}) = 3|x_{p+1},\ldots,x_{2p}|+1$.

case f. Otherwise. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}) = 2|x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}|+1$.

Note that in case 1, $|x_1,\ldots,x_q,y| = |x_1,\ldots,x_q|$. Also note that in cases a)-e),

$$|x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}| = |y_1,\ldots,y_{q+1}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}| = |x_{4p+1},\ldots,x_{5p}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}| = |x_{3p+1},\ldots,x_{4p}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}| = |x_{2p+1},\ldots,x_{3p}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_{q+1}| = |x_{p+1},\ldots,x_{2p}|$$

respectively. Hence $f',g' \in ELG \cap SD \cap BAF$.

Let $S \subseteq N$. From S, case 1 produces exactly $g(S^*)$. Case 2 produces exactly $6S+2$. This establishes i).

Case a) produces exactly $g'S$. Case b) produces exactly $f(S^*)$. Case c) produces exactly $6f(S^*)+2$. Case d produces exactly $2S^*+1$. Case e produces exactly $3S^*+1$.

Case f) produces exactly $2S^*+1$ since $2\min(S)+1$ is not produced. This is because $2\min(S)+1$ can only be produced from case f) if all of the arguments are $\min(S)$, which can only happen under case a). This establishes ii). QED

LEMMA 5.1.7. Let $f,g \in ELG \cap SD \cap BAF$ and $rng(g) \subseteq 6N$. There exist infinite $A \subseteq B \subseteq C \subseteq N\backslash\{0\}$ such that
i) $fA \cap 6N \subseteq B \cup gB$;
ii) $fB \cap 6N \subseteq C \cup gC$;
iii) $fA \cap 2N+1 \subseteq B$;
iv) $fA \cap 3N+1 \subseteq B$;
v) $fB \cap 2N+1 \subseteq C$;
vi) $fB \cap 3N+1 \subseteq C$;
vii) $C \cap gC = \emptyset$;
viii) $A \cap fB = \emptyset$.

Proof: Let f,g be as given. Let f′,g′ be given by Lemma 5.1.6. Let $A,B,C \subseteq N$ be given by Proposition C for f′,g′. We have

$$A \cup. f'A \subseteq C \cup. g'B$$
$$A \cup. f'B \subseteq C \cup. g'C.$$

Let $n \in B$. Then $6n+2 \in g'B \subseteq f'B$, and so $6n+2 \in C \vee 6n+2 \in g'C$. Now $6n+2 \notin C$ by $C \cap g'B = \emptyset$. Hence $6n+2 \in g'C$. By Lemma 5.1.6 i) and $rng(g) \subseteq 6N$, we have $6n+2 \in 6C+2$. Therefore $n \in C$. So we have established that $B \subseteq C$.

Let $n \in A$. Then $n \in C \vee n \in g'B$. Now $n \notin f'B$ by $A \cap f'B = \emptyset$. Also $g'B \subseteq f'B$. Hence $n \notin g'B$, $n \in C$. Also $6n+2 \in g'A \subseteq f'A$, and so $6n+2 \in C \vee 6n+2 \in g'B$. Since $n \in C$, we have $6n+2 \in g'C$. By $C \cap g'C = \emptyset$, we have $6n+2 \notin C$. Hence $6n+2 \in g'B$. Since $rng(g) \subseteq 6N$, we have $6n+2 \in 6B+2$. Hence $n \in B$. So we have established that $A \subseteq B$.

We have thus shown that $A \subseteq B \subseteq C \subseteq N$.

We now verify all of the required conditions i)-viii) above using the three sets A^*,B^*,C^*.

Firstly note that $A^* \subseteq B^* \subseteq C^* \subseteq N\backslash\{0\}$. To see this, let $n \in A^*$. Then $n \in A \wedge n > \min(A)$. Hence $n \in B \wedge n > \min(B)$, and so $n \in B^*$. By the same argument, $n \in B^* \rightarrow n \in C^*$.

We now claim that $A^* \cap f(B^*) = \emptyset$. This follows from $A^* \subseteq A$ and $f(B^*) \subseteq f'B$.

Next we claim that $C^* \cap g(C^*) = \emptyset$. This follows from $C^* \subseteq C$ and $g(C^*) \subseteq g'C$.

Now we claim that $f(A^*) \cap 6N \subseteq B^* \cup g(B^*)$. To see this, let $n \in f(A^*) \cap 6N$. Then $n \in f'A$. Hence $n \in C \cup g'B$.

case 1. $n \in C$. Now $6n+2 \in g'C$ and $6n+2 \in 6f(A^*)+2 \subseteq f'A$. Since $C \cap g'C = \emptyset$, we have $6n+2 \notin C$. Also $6n+2 \in C \cup g'B$. Hence $6n+2 \in g'B$. Since $rng(g) \subseteq 6N$, we have $6n+2 \in 6B+2$, and so $n \in B$. Since $n \in f(A^*)$ and f is strictly dominating, we have $n > min(A) \geq min(B)$. Hence $n \in B^*$.

case 2. $n \in g'B$. Since $n \in 6N$, $n \in g(B^*)$. This establishes the claim.

Next we claim that $f(B^*) \cap 6N \subseteq C^* \cup g(C^*)$. To see this, let $n \in f(B^*) \cap 6N$. Then $n \in f'B$. Hence $n \in C \cup g'C$.

case 1'. $n \in C$. Since $n \in f(B^*)$ and f is strictly dominating, we have $n > min(B) \geq min(C)$. Hence $n \in C^*$.

case 2'. $n \in g'C$. Since $n \in 6N$, $n \in g(C^*)$. This establishes the claim.

Now we claim that $f(A^*) \cap 2N+1$, $f(A^*) \cap 3N+1 \subseteq B^*$. To see this, let $n \in f(A^*)$, $n \in 2N+1 \cup 3N+1$. Then $n \in f'A$, and so $n \in C \cup g'B$. Recall that $rng(g) \subseteq 6N$. Since $n \in 2N+1 \cup 3N+1$, we see that $n \notin g'B$, and so $n \in C$. Now $6n+2 \in g'C$ and $6n+2 \in 6f(A^*)+2 \subseteq f'A$. Since $C \cap g'C = \emptyset$, we have $6n+2 \notin C$. Also $6n+2 \in f'A \subseteq C \cup g'B$. Hence $6n+2 \in g'B$. Since $rng(g) \subseteq 6N$, we have $6n+2 \in 6B+2$, and so $n \in B$. Since $n \in f(A^*)$ and f is strictly dominating on A, we have $n > min(A) \geq min(B)$. Hence $n \in B^*$.

Finally we claim that $f(B^*) \cap 2N+1$, $f(B^*) \cap 3N+1 \subseteq C^*$. To see this, let $n \in f(B^*)$, $n \in 2N+1 \cup 3N+1$. Then $n \in f'B$, and so $n \in C \cup g'C$. Since $n \in 2N+1 \cup 3N+1$, we have $n \notin 6N \cup 6N+2$. Hence $n \notin g'C$, $n \in C$. Since $n \in f(B^*)$ and f is strictly dominating, $n > min(B) \geq min(C)$. Hence $n \in C^*$. QED

The phrase "length 3 towers" mentioned in the title of this section refers to the $A \subseteq B \subseteq C$ in Lemma 5.1.7.