5.3. Countable nonstandard models with limited indiscernibles.

LEMMA 5.3.1. There exist positive integers $\sigma_1,\tau_1,\sigma_2,\tau_2,...$, each divisible by 96, such that for all $n \geq 1$ and $x,y \in N$,

$$\sigma_n x + \tau_n = \sigma_m y + \tau_m \rightarrow (n = m \wedge x = y).$$
$$\sigma_n,\tau_n \geq 96n.$$

Proof: For $n \geq 1$, let $\sigma_n = 96(p_n!)$ and $\tau_n = 96p_n$, where p_n is the n-th prime. Suppose $96(p_n!)x + 96p_n = 96(p_m!)y + 96p_m$. Then $p_n!x + p_n = p_m!y + p_m$. If $n \leq m$ then p_n clearly divides the left side and the first term of the right side. Hence p_n divides p_m. Therefore $n = m$. If $m \leq n$ then also $n = m$. Hence $n = m$. Therefore $x = y$. QED

DEFINITION 5.3.1. We fix σ_n,τ_n, $n \geq 1$, as given by Lemma 5.3.1.

Recall the standard pairing function P (Definition 3.2.1). We have $P(n,m) \geq n,m$. We use the extension $P(x,y,z) = P(P(x,y),z)$. We have $P(n,m,r) \geq n,m,r$, and P is strictly increasing in each argument.

The following Lemma adjoins r predicates $E_1,...,E_r \subseteq N$ to our basic standard countable structure $(N,<,0,1,+,-,\bullet,\uparrow,\log)$.

LEMMA 5.3.2. Let $r \geq 3$. There exists a structure $(N,<,0,1,+,-,\bullet,\uparrow,\log,E_1,...,E_r)$ such that the following holds.
i) $E_1 \subseteq ... \subseteq E_r \subseteq N\backslash\{0\}$;
ii) $|E_1| = r$ and E_r is finite;
iii) For all $x < y$ from E_1, $x\uparrow < y$;
iv) Let $1 \leq i \leq \gamma(r)$, $1 \leq j < r$, $0 \leq a,b < r$, and $x \in \alpha(r,E_j;1,r)$. Then $(\exists v_2,...,v_r \in E_{j+1})(v_2,...,v_r \leq ax+b \wedge \varphi[i,r](x,v_2,...,v_r)) \leftrightarrow \sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \notin E_{j+1}$;
v) For all $1 \leq j \leq r-1$, $2\alpha(r,E_j;1,r)+1$, $3\alpha(r,E_j;1,r)+1 \subseteq E_{j+1}$;
vi) $E_1 \cap \alpha(r,E_2;2,r) = \emptyset$;
vii) Let $1 \leq i \leq \beta(2r)$, $x_1,...,x_{2p} \in E_1$, $y_1,...,y_r \in \alpha(r,E_2)$, where $(x_1,...,x_r)$ and $(x_{r+1},...,x_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(x_1,...,x_r)$. Then $t[i,2r](x_1,...,x_r,y_1,...,y_r) \in E_3 \leftrightarrow t[i,2r](x_{r+1},...,x_{2r},y_1,...,y_r) \in E_3$.

Proof: Let $r \geq 3$, and let $r' \gg r$. We define $\gamma(r)+3r$-ary $g \in$ BAF with $rng(g) \subseteq 48N$, as follows. Let $x\# = (y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_r,w_1,\ldots,w_r,x,x_2,\ldots,x_r) \in N^{\gamma(r)+3r}$. Let i be greatest such that $y_1 = \ldots = y_{i+1}$. Let a be greatest such that $z_1 = \ldots = z_{a+1}$. Let b be greatest such that $w_1 = \ldots = w_{b+1}$. (It will prove to be convenient to write x here instead of x_1.)

case 1. $0 < |x\#| \leq (3+a+b)x \wedge |x_2,\ldots,x_r| \leq ax+b \wedge \varphi[i,r](x,x_2,\ldots,x_r)$. Set $g(x\#) = \sigma_{P(i,a,b)}x+\tau_{P(i,a,b)}$.

case 2. Otherwise. Set $g(x\#) = 96|x\#|+48$.

In case 1, $g(x\#) \geq 96P(i,a,b)x \geq 96P(1,a,b)x \geq 96\max(1,a,b)x \geq 32(1+a+b)x \geq 8(3+a+b)x \geq 8|x\#| > |x\#|$. Also $g(x\#) \leq \sigma_{P(i,a,b)}|x\#|+\tau_{P(i,a,b)}$
$\leq (\sigma_{P(i,a,b)}+\tau_{P(i,a,b)})|x\#|$.

In case 2, $|x\#| < g(x\#) \leq 192|x\#|$. Hence $g \in ELG \cap SD \cap BAF$.

Clearly $rng(g) \subseteq 48N$. So we can apply Lemma 5.2.12 with r'. Let $D_1,\ldots,D_{r'} \subseteq N$ be given by Lemma 5.2.12. For all $1 < i \leq r$, set $E_i = D_i$. Set E_1 to be the first r elements of D_1.

Claims i),ii),iii),v),vi) for $E_1,\ldots,E_r$ follow immediately from clauses i),ii),iii),v),vii) for $D_1,\ldots,D_{r'}$ in Lemma 5.2.12.

For claim iv), let $1 \leq i \leq \gamma(r)$, $1 \leq j < r$, $0 \leq a,b < r$, and $x \in \alpha(r,E_j;1,r)$. We claim that

$$\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \in 48\alpha(r',E_j;1,r').$$

To see this, write

$$|d_1,\ldots,d_k| \leq x = t[i,r](d_1,\ldots,d_k) \leq r|d_1,\ldots,d_k|,$$

where $k \geq 1$ and $d_1,\ldots,d_k \in E_j$. Since $96|\sigma_{P(i,a,b)}, 96|\tau_{P(i,a,b)}$, let $p = \sigma_{P(i,a,b)}/48$ and $q = \tau_{P(i,a,b)}/48$. Then we have

$$px+q \in \alpha(r',E_j;1,r')$$

since $r' \gg r$ and $p,q > 0$. This establishes the claim.

Since $r' \geq r$, by Lemma 5.2.12 iv),vi),

$$\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \in E_{j+1} \cup.\ gE_{j+1}.$$

First assume that

$$\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \notin E_{j+1}.$$

Then $\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \in gE_{j+1}$. Write

$$\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} = g(y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_r,w_1,\ldots,w_r,u,u_2,\ldots,u_r),$$

where $y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_r,w_1,\ldots,w_r,u,u_2,\ldots,u_r \in E_{j+1}$.

Since 96 divides $\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)}$, $\sigma_{(i,a,b,)}x+\tau_{P(i,a,b)}$ can only arise from case 1 in the definition of g.

Let i′ be greatest such that $y_1 = \ldots = y_{i'+1}$, a′ be greatest such that $z_1 = \ldots = z_{a'+1}$, and b′ be greatest such that $w_1 = \ldots = w_{b'+1}$. Then

$$0 < |y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_r,w_1,\ldots,w_r,u,u_2,\ldots,u_r| \leq (3+a'+b')x.$$
$$u_2,\ldots,u_r \leq a'u+b'.$$
$$\varphi[i',r](u,u_2,\ldots,u_r).$$
$$g(y_1,\ldots,y_{\gamma(r)},z_1,\ldots,z_r,w_1,\ldots,w_r,u,u_2,\ldots,u_r) = \sigma_{P(i',a',b')}u + \tau_{P(i',a',b')} = \sigma_{P(i,a,b)}x + \tau_{P(i,a,b)}.$$

By Lemma 5.3.1,

$$i = i' \wedge a = a' \wedge \wedge b = b' \wedge u = x.$$

Hence

$$u_2,\ldots,u_r \leq ax+b.$$
$$\varphi[i,r](x,u_2,\ldots,u_r).$$

In particular,

$$(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi[i,r](x,v_2,\ldots,v_r)).$$

Now assume that

$$(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi[i,r](x,v_2,\ldots,v_r)).$$

Let

$$x_2,\ldots,x_r \in E_{j+1}.$$
$$x_2,\ldots,x_r \le ax+b.$$
$$\varphi[i,r](x,x_2,\ldots,x_r).$$

By Lemma 5.2.12 v), $2x+1 \in 2\alpha(r';E_j,1,r')+1 \subseteq D_{j+1} = E_{j+1}$. Note that

$$0 < |x,\ldots,x,2x+1,\ldots,2x+1,x,\ldots,x,2x+1,\ldots,2x+1, x,\ldots,x,2x+1,\ldots,2x+1,x,x_2,\ldots,x_r| \le \max(2x+1,ax+b) \le (3+a+b)x.$$
$$x_2,\ldots,x_r \le ax+b.$$
$$\varphi[i,r](x,x_2,\ldots,x_r).$$

Here the first group of x's has length i, the second group of x's has length a, and the third group of x's has length b.

Hence case 1 applies, and so

$$g(x,\ldots,x,2x+1,\ldots,2x+1,x,\ldots,x,2x+1,\ldots,2x+1, x,\ldots,x,2x+1,\ldots,2x+1,x,x_2,\ldots,x_r) = \sigma_{P(i,a,b)}x+\tau_{P(i,a,b)}.$$

Hence $\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \in gE_{j+1}$. By Lemma 5.2.12, vi), $E_{j+1} \cap gE_{j+1} = \emptyset$. Hence $\sigma_{P(i,a,b)}x+\tau_{P(i,a,b)} \notin E_{j+1}$ as required. This establishes claim iv).

Claim vii) is immediate from Lemma 5.2.12 vii). QED

We now work with countable structures whose domain is not N. These structures must interpret the language L, so that we work with structures of the form (A,<,0,1,+,-,•,log,...). In fact, this is why we wrote (N,<,0,1,+,-,•,↑,log,$E_1,\ldots,E_r$) in Lemma 5.3.2.

Let M = (A,<,0,1,+,-,•,↑,log,...) be given. In all such M that we consider, M will satisfy a certain amount of arithmetic. In particular, M will satisfy TR(Π^0_1,L). See the discussion below.

DEFINITION 5.3.2. Let $E \subseteq A$ and $p \in N$. We write $\alpha(E;p,<\infty)$ for the set of $x \in A$ such that the following holds. There is a term t in L that is not closed, and an assignment f to its variables, with rng(f) $\subseteq$ E, such that x is the value of t under f, and a nonnegative integer k, such that $x \in$ [pmax(rng(f)),kmax(rng(f))].

In the above, the value of t under f is computed using the interpretation M of L. It is important to note that here both p and k serve as standard integers.

DEFINITION 5.3.3. We let $\alpha(E)$ be the set of all values of terms t in L under an assignment of elements of E to the variables in t, computed according to M. For $\alpha(E)$, we allow closed terms.

DEFINITION 5.3.4. We also let $\alpha(r,E)$ be the set of all values of terms t in L with $\#(t) \leq r$, under an assignment of elements of E to the variables in t, computed according to M. For $\alpha(r,E)$, we also allow closed terms.

Recall the theory $TR(\Pi^0_1,L)$ from Definition 5.1.10. It is clear that $(N,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies PA(L) + $TR(\Pi^0_1,L)$, where PA(L) is Peano Arithmetic, formulated in L. See Definition 5.6.6.

LEMMA 5.3.3. There exists a countable structure M = $(A,<,0,1,+,-,\bullet,\uparrow,\log,E_1,E_2,...)$ obeying the following conditions.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0_1,L)$;
ii) $E_1 \subseteq E_2 \subseteq ... \subseteq A\backslash\{0\}$;
iii) E_1 has order type ω, and has no upper bound in A;
iv) For all $x < y$ from E_1, $x\uparrow < y$;
v) Let $r \geq 1$, $\varphi(v_1,...,v_r)$ be a quantifier free formula of L, and $a,b \in N$. There exists $d,e \in N\backslash\{0\}$ such that for all $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$, $(\exists v_2,...,v_r \in E_{j+1})(v_2,...,v_r \leq ax+b \wedge \varphi(x,v_2,...,v_r)) \leftrightarrow dx+e \notin E_{j+1}$;
vi) For all $j \geq 1$, $2\alpha(E_j;1,<\infty)+1$, $3\alpha(E_j;1,<\infty)+1 \subseteq E_{j+1}$;
vii) $E_1 \cap \alpha(E_2;2,<\infty) = \emptyset$;
viii) Let $r \geq 1$ and $t(v_1,...,v_{2r})$ be a term of L. Let $x_1,...,x_{2r} \in E_1$, $y_1,...,y_r \in \alpha(E_2)$, where $(x_1,...,x_r)$ and $(x_{r+1},...,x_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(x_1,...,x_r)$. Then $t(x_1,...,x_r,y_1,...,y_r) \in E_3 \leftrightarrow t(x_{r+1},...,x_{2r},y_1,...,y_r) \in E_3$.

Proof: We apply the compactness theorem for predicate calculus with equality, to Lemma 5.3.2.

For the purposes of the proof of this Lemma only, we use the language L' which augments the language L with infinitely many unary relation symbols E_i, $i \geq 1$, and infinitely many constant symbols, c_i, $i \geq 1$.

Let T be the following set of axioms in L'.

1. $TR(\Pi^0{}_1,L)$.
2. For all $i \geq 1$, we take $c_i < c_{i+1} \wedge c_i \in E_1$.
3. For all $i \geq 1$, we take $E_i \subseteq E_{i+1} \wedge 0 \notin E_i$.
4. $(\forall x,y \in E_1)(x < y \rightarrow x\uparrow < y)$.
5. Let $1 \leq i \leq \gamma(r)$, $1 \leq j < r$, and $0 \leq a,b < r$. $(\forall v_1 \in \alpha(E_j;1,<\infty))((\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq av_1+b \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow \sigma_{P(i,a,b)}v_1+\tau_{P(i,a,b)} \notin E_{j+1})$. Here we index over r,i,j,a,b, and also non closed terms and standard integer upper coefficients for $\alpha(E_j;1,<\infty)$. The latter is used to create infinitely many conditional members of $\alpha(E_j;1,<\infty)$).
6. Let $j \geq 1$. We take the schemes $2\alpha(E_j;1,<\infty)+1 \subseteq E_{j+1}$, $3\alpha(E_j;1,<\infty)+1 \subseteq E_{j+1}$. Here we index over j, and also non closed terms and standard integer upper coefficients for $\alpha(E_j;1,<\infty)$. The latter is used to create infinitely many conditional members of $\alpha(E_j;1,<\infty)$.
7. We take the scheme $E_1 \cap \alpha(E_2;2,<\infty) = \emptyset$. Here we index over non closed terms and also standard integer upper coefficients for $\alpha(E_2;2,<\infty)$. The latter is used to create infinitely many conditional members of $\alpha(E_j;2,<\infty)$.
8. Let $r \geq 1$ and t be a term in L with at most the variables $v_1,\ldots,v_{2r}$. Let $x_1,\ldots,x_{2r} \in E_1$, $y_1,\ldots,y_r \in \alpha(r,E_2)$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. Then $t(x_1,\ldots,x_r,y_1,\ldots,y_r) \in E_3 \leftrightarrow t(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in E_3$. Here we index over r,t, and terms for $\alpha(r,E_2)$. The latter is used to create finitely many conditional elements of $\alpha(r,E_2)$.

These axioms have very robust formulations in light of axioms 1. It suffices to remark that in 5, we take av+b to be

$$(1+\ldots+1)\bullet v_i+(1+\ldots+1)$$

where the 1's are associated to the left, and where there are a 1's in the first sum, and b 1's in the second sum. We take 0 1's to mean 0.

Let $T_0 \subseteq T$ be finite. Let $s \geq 3$ be so large that all of the i,r,j+1, values of β at # of terms, and upper coefficients used, are at most s.

Let $M_s = (N,<,0,1,+,-,\bullet,\uparrow,\log,E_1,\ldots,E_s)$ be given by Lemma 5.3.2, with r = s. We now show that M_s satisfies $T_0 + TR(\Pi^0{}_1,L)$, where for all $1 \leq i \leq s$, c_i is interpreted as the i-th element of E_1 (c_1 is interpreted as $\min(E_1)$). Obviously, M_s satisfies 1,2 of T_0 by construction.

The axioms in T_0 from 3-4,6,7 obviously hold in M_s using Lemma 5.3.2 i)-iii),v),vi).

For the axioms in T_0 from 5, we have to handle several different r's at once. This is because of the different lengths of the existential quantifiers that appear in 5.

We use our convenient coding setup whereby if $1 \leq i \leq \gamma(r)$ and $1 \leq r \leq s$, then $\varphi[i,r] = \varphi[i,s]$.

For 5, let $1 \leq i \leq \gamma(r)$, $1 \leq j < r$, and $0 \leq a,b < r$. The axioms in T_0 from 5 must have $r \leq s$. By Lemma 5.3.2 iv), M_s satisfies

$$(\forall v_1 \in \alpha(s,E_j;1,s))((\exists v_2,\ldots,v_s \in E_{j+1})(v_2,\ldots,v_s \leq av_1+b \wedge \varphi[i,s](v_1,\ldots,v_s)) \leftrightarrow \sigma_{P(i,a,b,)}v_1+\tau_{P(i,a,b)} \notin E_{j+1}).$$

It is clear that M_s satisfies

$$(\forall v_1 \in \alpha(r,E_j;1,r))((\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq av_1+b \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow \sigma_{P(i,a,b)}v_1+\tau_{P(i,a,b)} \notin E_{j+1})$$

since $1 \leq r \leq s$. In particular, all variables in $\varphi[i,s] = \varphi[i,r]$ are among $v_1,\ldots,v_r$, so the extra existential quantifiers, $v_{r+1},\ldots,v_s$, are dummy quantifiers. Therefore M_s satisfies the axioms in T_0 from 5.

We now come to the verification of 8 in M_s. Let $r \geq 1$, t be a term in L with at most the variables v1,...,v2r, $x_1,\ldots,x_{2r} \in E_1$, $y_1,\ldots,y_r \in \alpha(r,E_2)$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$.

The axioms in T_0 from 8 must have $r \leq s$. Also we can let $1 \leq i \leq \beta(2s)$ be such that t[i,2s] is the result of replacing the variables $v_{r+1},\ldots,v_{2r}$ in t by the variables $v_{s+1},\ldots,v_{s+r}$. By Lemma 5.3.2 vii), M_s satisfies

$$t[j,2s](x_1,\ldots,x_r,\ldots,x_r,y_1,\ldots,y_r,\ldots,y_r) \in E_3 \leftrightarrow t[j,2s](x_{r+1},\ldots,x_{2r},\ldots,x_{2r},y_1,\ldots,y_r,\ldots,y_r) \in E_3.$$

Note that

$$t[j,2s](x_1,\ldots,x_r,\ldots,x_r,y_1,\ldots,y_r,\ldots,y_r) = t(x_1,\ldots,x_r,y_1,\ldots,y_r).$$
$$t[j,2s](x_{r+1},\ldots,x_{2r},\ldots,x_{2r},y_1,\ldots,y_r,\ldots,y_r) = t(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r).$$

Hence M_s satisfies

$$t(x_1,\ldots,x_r,y_1,\ldots,y_r) \in E_3 \leftrightarrow t(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in E_3.$$

By the compactness theorem for first order predicate calculus with equality, T has a countable model M = $(A,<,0,1,+,-,\bullet,\uparrow,\log,E_1,E_2,\ldots)$. We now verify clauses i)-viii) of Lemma 5.3.3, except for clause iii). In order to verify clause iii), we must adjust M.

Claim i) is immediate from axioms 1 of T.

Claim ii) is immediate from axioms 3 of T.

Claim iv) is immediate from axioms 4 of T.

For claim v), let $r \geq 1$, $\varphi(x_1,\ldots,x_r)$ be a quantifier free formula of L, $a,b \in N$. By axiom 5 of T, M satisfies

$$(\forall v_1 \in \alpha(E_j;1,<\infty))$$
$$((\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax_1+b \wedge \varphi[i,r](v_1,\ldots,v_r)) \leftrightarrow \sigma_{P(i,a,b)}v_1+\tau_{P(i,a,b)} \notin E_{j+1}).$$

Set $d = \sigma_{P(i,a,b)}$ and $e = \tau_{P(i,a,b)}$.

For claim vi), let $r,j \geq 1$. By axiom 6 of T, M satisfies

$$2\alpha(r,E_j,1,r)+1 \subseteq E_{j+1}.$$
$$3\alpha(r,E_j,1,r)+1 \subseteq E_{j+1}.$$

Since r is arbitrary, M satisfies

$$2\alpha(E_j;1,<\infty)+1 \subseteq E_{j+1}.$$
$$3\alpha(E_j;1,<\infty)+1 \subseteq E_{j+1}.$$

Claim vii) follows immediately from axioms 7 of T.

Claim viii) also follows immediately from axioms 8 of T.

Now M may not satisfy iii). We will instead use an initial segment of M so that iii) holds. We need to check that the above verifications in M are still valid in our initial segment of M (defined below).

By axioms 2 of T, let $E_1' \subseteq E_1$ be of order type ω. Let

$$A' = \{x \in \mathrm{dom}(M): (\exists y \in E_1')(x < y)\}.$$

Note that by axioms 1,4 of T, A' is closed under all of the primitive operations of L. Hence we let M' be M restricted to A', where the E_1 of M' is E_1', and the E_j of M', $j \geq 2$, is $E_j \cap A'$.

We now show that M' satisfies all of the claims i)-viii).

For i), note that M' is still a model of $TR(\Pi^0_1,L)$ because M' is an initial segment of M that is closed under the operations of M.

Claims ii),iii) are immediate by the definitions of the E_i of M'.

The remaining claims are all immediate since all of the quantifiers are bounded, the initial segment A' is closed under all of the primitive operations of M, and E_1 has been shrunk to $E_1' \subseteq E_1$. QED

We fix $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E_1,E_2,...)$ as given by Lemma 5.3.3.

DEFINITION 5.3.5. We will use the notation $|y_1,...,y_r|$ for the maximum of $y_1,...,y_r \in A$ in the sense of M.

LEMMA 5.3.4. Every element of E_1 is nonstandard.

Proof: Fix a standard element k of E_1. Let t(x) be a term of L such that

$$t(x) = k \text{ if } x = k;\ 0 \text{ otherwise.}$$

By Lemma 5.3.3 iii), let $k < n$, $n \in E_1$. By Lemma 5.3.3 viii),

$$t(k) \in E_3 \leftrightarrow t(n) \in E_3.$$
$$k \in E_3 \leftrightarrow 0 \in E_3.$$

This is a contradiction since $k \in E_3$ and $0 \notin E_3$, by Lemma 5.3.3 ii). QED

LEMMA 5.3.5. Let $r,j \geq 1$, $\varphi(v_1,\ldots,v_r)$ be a quantifier free formula of L, $a,b \in N$, and $x \in \alpha(E_1;1,<\infty)$ Then $(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r)) \leftrightarrow (\exists v_2,\ldots,v_r \in E_2)(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r))$.

Proof: Let $r \geq 1$, $\varphi(v_1,\ldots,v_r)$ be a quantifier free formula of L, and $a,b \in N$. By Lemma 5.3.3 v), let $d,e \in N\backslash\{0\}$ be such that the following holds. For all $j \geq 1$ and $x_1 \in \alpha(E_1;1,<\infty)$,

$$(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r)) \leftrightarrow dx+e \notin E_{j+1}.$$

Now let $j \geq 1$ and $x \in \alpha(E_1;1,<\infty)$. Then

$$(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r)) \rightarrow dx+e \notin E_{j+1} \rightarrow dx+e \notin E_2 \rightarrow (\exists v_2,\ldots,v_r \in E_2)(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r)).$$

QED

LEMMA 5.3.6. For all $j \geq 1$, $E_1 \cap \alpha(E_j;2,<\infty) = \emptyset$.

Proof: By Lemma 5.3.3 vii), this is true for $j = 1,2$. Suppose this is false for some fixed $j \geq 3$. Let $y \in E_1$, $p,r \geq 1$, $t(v_2,\ldots,v_r)$ be a term of L, $x_2,\ldots,x_r \in E_j$, $y = t(x_2,\ldots,x_r)$, and $2|x_2,\ldots,x_r| \leq y \leq p|x_2,\ldots,x_k|$. Then

$$(\exists v_2,\ldots,v_r \in E_j)(v_2,\ldots,v_r \leq y \wedge 2|v_2,\ldots,v_r| \leq y \leq p|v_1,\ldots,v_r| \wedge y = t(v_2,\ldots,v_r)).$$

By Lemma 5.3.5,

$$(\exists v_2,\ldots,v_r \in E_2)(v_2,\ldots,v_r \leq y \wedge 2|v_2,\ldots,v_r| \leq y \leq p|v_2,\ldots,v_r| \wedge y = t(v_2,\ldots,v_r)).$$

Therefore $y \in \alpha(E_2;2,<\infty)$. Since $y \in E_1$, this contradicts Lemma 5.3.3 vii). QED

LEMMA 5.3.7. Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a quantifier free formula of L. Let $x_1,\ldots,x_{2r} \in E_1$, $y_1,\ldots,y_r \in \alpha(E_2)$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. Then $\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow \varphi(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r)$.

Proof: Let r,φ be as given. Let $f:N^{2r} \to N$ be defined by

$$f(x_1,\ldots,x_{2r}) = 0 \text{ if } \varphi(x_1,\ldots,x_{2r});\ x_1 \text{ otherwise.}$$

Obviously, f is given by a term $t(x_1,\ldots,x_{2r})$ of L.

Let $x_1,\ldots,x_{2r},y_1,\ldots,y_r$ be as given. Since $0 \notin E_3$ and $x_1,\ldots,x_{2r} \in E_1$, we have

$$\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow t(x_1,\ldots,x_r,y_1,\ldots,y_r) \notin E_3.$$
$$\varphi(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \leftrightarrow t(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \notin E_3.$$

By Lemma 5.3.3 viii),

$$t(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in E_3 \leftrightarrow t(x_1,\ldots,x_r,y_1,\ldots,y_r) \in E_3.$$
$$\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow \varphi(x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r).$$

QED

DEFINITION 5.3.6. For x in A and $k \geq 0$, we write $\uparrow k(x)$ for $x\uparrow\ldots\uparrow$, where there are k $\uparrow$'s. We take $\uparrow 0(x) = x$.

DEFINITION 5.3.7. By Lemma 5.3.4 and Lemma 5.3.3 iii), we fix $c_1 < c_2 < \ldots$ to be the strictly increasing ω enumeration of E_1, which consists entirely of nonstandard elements. The c's are unbounded in A.

LEMMA 5.3.8. Let $r \geq 1$ and $t(v_1,\ldots,v_r)$ be a term of L. There exists $p \in N$ such that for all $x_1,\ldots,x_r$, $t(x_1,\ldots,x_r) \leq \uparrow p(|x_1,\ldots,x_r|)$. Furthermore, if $x_1,\ldots,x_r \leq c_n$ then $t(x_1,\ldots,x_r) < c_{n+1}$.

Proof: Let $r \geq 1$. The first claim is easily proved by induction on the term $t(v_1,\ldots,v_r)$.

We now show that for all $p \in N$ and $n \geq 1$, $\uparrow p(c_n) < c_{n+1}$. Suppose $\uparrow p(c_n) \geq c_{n+1}$. By Lemma 5.3.7, for all $m \geq n+1$, $\uparrow p(c_n) \geq c_m$. But this contradicts Lemma 5.3.3 iii), that the c's have no upper bound in A.

For the second claim, we use p from the first claim, which depends only on r,t. Let $x_1,\ldots,x_r \leq c_n$. Then $t(x_1,\ldots,x_r) \leq \uparrow p(|x_1,\ldots,x_r|) \leq \uparrow p(c_n) < c_{n+1}$. QED

LEMMA 5.3.9. For all $a,b \in N$ there exist $c,d \in N\backslash\{0\}$ such that the following holds. Let $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$. Then $ax+b \in E_{j+1} \leftrightarrow cx+d \notin E_{j+1}$.

Proof: Let $a,b \in N$. By Lemma 5.3.3 v), let $c,d \in N\backslash\{0\}$ be such that the following holds. Let $j \geq 1$ and $x \in E_j$. Then

$$(\exists v_2 \in E_{j+1})(v_2 \leq ax+b \wedge v_2 = ax+b) \leftrightarrow cx+d \notin E_{j+1}.$$

I.e.,

$$ax+b \in E_{j+1} \leftrightarrow cx+d \notin E_{j+1}.$$

QED

LEMMA 5.3.10. Let $r \geq 1$, $a,b \in N$, and $\varphi(x_1,\ldots,x_r)$ be a quantifier free formula in L. There exist $d,e,f,g \in N\backslash\{0\}$ such that the following holds. Let $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$. Then $(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r)) \leftrightarrow dx+e \notin E_{j+1} \leftrightarrow fx+g \in E_{j+1}$.

Proof: Let r,φ,a,b be as given. By Lemma 5.3.3 v), let $d,e \in N\backslash\{0\}$ be such that the following holds. Let $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$. Then

$$(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi(x,v_2,\ldots,v_r)) \leftrightarrow dx+e \notin E_{j+1}.$$

By Lemma 5.3.9, let $f,g \in N\backslash\{0\}$ be such that for all $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$,

$$dx+e \in E_{j+1} \leftrightarrow fx+g \notin E_{j+1}.$$

$$(\exists v_2,\ldots,v_r \in E_{j+1})(v_2,\ldots,v_r \leq ax+b \wedge \varphi(v_1,\ldots,v_r)) \leftrightarrow dx+e \notin E_{j+1} \leftrightarrow px+q \in E_{j+1}.$$

QED

We view Lemma 5.3.10 as a form of quantifier elimination without parameters. We need to develop a workable form of this kind of quantifier elimination with parameters. This requires that we have a mechanism for coding up tuples.

DEFINITION 5.3.8. We adapt the standard pairing function of Definition 3.2.1 to map A^2 onto A, for our structure M. I.e., for all $x,y \in A$,

$$P(x,y) = (x+y)(x+y+1)/2 + y = (x^2+y^2+2xy+x+3y)/2.$$

By Lemma 5.3.3 i, M satisfies $TR(\Pi^0{}_1,L)$. Hence $P:A^2 \to A$ is one-one, onto, P is strictly increasing in each argument, and for all $x,y \in A$, $P(x,y) \geq x,y$.

DEFINITION 5.3.9. We extend P naturally to any finite number of arguments by $P(x) = x$, and for $k \geq 3$, $P(x_1,\ldots,x_k) = P(P(x_1,x_2),x_3,\ldots,x_k)$.

Note that for each $k \geq 1$, P is a bijection from A^k onto A, P is strictly increasing in each argument, and for all $x_1,\ldots,x_k \in A$, $P(x_1,\ldots,x_k) \geq x_1,\ldots,x_k$.

DEFINITION 5.3.10. We also define, in M,

$$x \div y = \text{the unique } z \text{ such that } y \bullet z \leq x < (y+1) \bullet z \text{ if } y \neq 0; \quad 0 \text{ otherwise.}$$

Let $x_1,\ldots,x_k \leq c_n$, where $x_1,\ldots,x_k \in E_j$. Suppose we want to code $x_1,\ldots,x_k$. The natural choice is of course $P(x_1,\ldots,x_k)$. However, at this point, we don't know that $P(x_1,\ldots,x_k) \in E_j$, or even $P(x_1,\ldots,x_k) \in E_{j+1}$, $2P(x_1,\ldots,x_k) \in E_{j+1}$, or $3P(x_1,\ldots,x_k)+1 \in E_{j+1}$, which severely limits the usefulness of $P(x_1,\ldots x_k)$.

Our approach is to use $c_m > c_n$ to give a code for $x_1,\ldots,x_k \leq c_n$ that we can really use. For each $m > n$, a useful code for $x_1,\ldots,x_k \leq c_n$ is

$$CODE(c_m;x_1,\ldots,x_k) = 8((\log(c_m))\uparrow+P(x_1,\ldots,x_k))+1.$$

We make a more general definition.

DEFINITION 5.3.11. We define

$$CODE(w;x_1,\ldots,x_k) = 8((\log(w))\uparrow+P(x_1,\ldots,x_k))+1.$$
$$INCODE(x) = (x-(\log(x))\uparrow-1)\div 8.$$

Here $\div 8$ is the floor of the result of dividing by 8. Also, the $-$ here is associated to the left: $a-b-c = (a-b)-c$.

Note that CODE is given by a term in L. However, INCODE (inverse code) is not given by a term in L. So we have to be careful how we use INCODE. This issue arises in the proof of Lemma 5.3.13, statement 1). But note that

$$y \div 8 = z \leftrightarrow 8z \leq y < 8z+8.$$
$$\mathrm{INCODE}(x) = z \leftrightarrow 8z \leq x-(\log(x))\uparrow-1 < 8z+8.$$

Thus the associated graphs are expressible as quantifier free formulas of L. This supports careful use of INCODE.

Recall that from the proof of Theorem 5.1.5, $\log(w)\uparrow$ is the greatest power of 2 that is $\leq w$ if $w > 0$; 1 otherwise.

LEMMA 5.3.11. Let $k,n,m \geq 1$, and $x_1,\ldots,x_k \leq c_n < c_m$, where $x_1,\ldots,x_k \in \alpha(E_j;1,<\infty)$. Then $\mathrm{CODE}(c_m;x_1,\ldots,x_k) \in \alpha(E_j;2,<\infty) \cap E_{j+1}$. Separately, let $k \geq 1$ and $8P(x_1,\ldots,x_k)+1 < \log(w)$. Then $\mathrm{INCODE}(\mathrm{CODE}(w;x_1,\ldots,x_k)) = P(x_1,\ldots,x_k)$.

Proof: Let $k,n,m,x_1,\ldots,x_k$ be as given. Note that

$$(c_m \div 2)+1 \leq (\log(c_m))\uparrow \leq c_m.$$
$$2c_m \leq 4(\log(c_m))\uparrow+P(x_1,\ldots x_k)) \leq 5c_m.$$

$$4((\log(c_m))\uparrow+P(x_1,\ldots x_k)) \in \alpha(E_j;2,<\infty).$$
$$\mathrm{CODE}(c_m;x_1,\ldots,x_k) \in 2\alpha(E_j;2,<\infty)+1.$$

We have $\mathrm{CODE}(c_m;x_1,\ldots,x_k) \in E_{j+1}$ by Lemma 5.3.3 vi).

Now let $k,x_1,\ldots,x_k,w$ be as given. We claim that

1) $\log(\mathrm{CODE}(w;x_1,\ldots,x_k)) = \log(w)+3.$

To see this, note that

$$\begin{gathered}\log(\mathrm{CODE}(w;x_1,\ldots,x_k)) = \\ \log(8((\log(w))\uparrow + P(x_1,\ldots,x_k))+1) = \\ \log(8(\log(w))\uparrow + 8P(x_1,\ldots,x_k) + 1) = \\ \log((\log(w)+3)\uparrow + 8P(x_1,\ldots,x_k) + 1) \leq \\ \log((\log(w)+3)\uparrow + \log(w)) = \log(w)+3 \leq \\ \log((\log(w)+3)\uparrow + 8P(x_1,\ldots,x_k) + 1).\end{gathered}$$

Using 1),

$$\begin{gathered}\mathrm{INCODE}(\mathrm{CODE}(w;x_1,\ldots,x_k)) = z \leftrightarrow \\ 8z \leq \mathrm{CODE}(w;x_1,\ldots,x_k)-(\log(\mathrm{CODE}(w;x_1,\ldots,x_k)))\uparrow-1 < 8z+8 \leftrightarrow \\ 8z \leq \mathrm{CODE}(w;x_1,\ldots,x_k)-(\log(w)+3)\uparrow-1 < 8z+8 \leftrightarrow \\ 8z \leq \mathrm{CODE}(w;x_1,\ldots,x_k)-8((\log(w))\uparrow)-1 < 8z+8 \leftrightarrow \\ 8z \leq 8P(x_1,\ldots,x_k) < 8z+8.\end{gathered}$$

Hence

$$\text{INCODE}(\text{CODE}(w;x_1,...,x_k)) = P(x_1,...,x_k).$$

QED

LEMMA 5.3.12. Let $x \in \alpha(E_j;1,<\infty)$. There exist $y,z \in E_{j+1} \cap [0,4x]$ such that $x = y-z$.

Proof: Let x be as given. By Lemma 5.3.3 vi), $2x+1$, $3x+1 \in E_{j+1}$. Write $x = (3x+1)-(2x+1)$. QED

LEMMA 5.3.13. Let $r \geq 1$, $p \geq 2$, and $\varphi(v_1,...,v_{2r})$ be a quantifier free formula of L. There exist $a,b,d,e \in N\setminus\{0\}$ such that the following holds. Let $j,n \geq 1$ and $x_1,...,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$. Then

$$(\exists v_{r+1},...,v_{2r} \in E_{j+1})(v_{r+1},...,v_{2r} \leq \uparrow p(|x_1,...,x_r|) \wedge \varphi(x_1,...,x_r,v_{r+1},...,v_{2r})) \leftrightarrow$$
$$a\text{CODE}(c_{n+1};x_1,...,x_r)+b \notin E_{j+1} \leftrightarrow$$
$$d\text{CODE}(c_{n+1};x_1,...,x_r)+e \in E_{j+1}.$$

Proof: Let r,p,φ be as given. By Lemma 5.3.10, let $a,b,d,e \in N\setminus\{0\}$ be such that the following holds. Let $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$. Then

1) $(\exists v_1,...,v_{3r} \in E_{j+1})(v_1,...,v_{3r} \leq x \wedge v_{r+1},...,v_{2r} \leq \uparrow p(|v_{2r+1}-v_1,...,v_{3r}-v_r|) \wedge \text{INCODE}(x) = P(v_{2r+1}-v_1,...,v_{3r}-v_r) \wedge \varphi(v_{2r+1}-v_1,...,v_{3r}-v_r,v_{r+1},...,v_{2r})) \leftrightarrow ax+b \notin E_{j+1} \leftrightarrow dx+e \in E_{j+1}$.

We have used the fact that $\text{INCODE}(v) = P(v_1,...,v_r)$ can be expanded out as a formula of L in variables $v,v_1,...,v_r$, as observed just before Lemma 5.3.11.

Now let $j,n \geq 1$, $x_1,...,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$. By Lemma 5.3.11, $\text{CODE}(c_{n+1};x_1,...,x_r) \in \alpha(E_j;1,<\infty)$. Hence we can set $x = \text{CODE}(c_{n+1};x_1,...,x_r)$ and obtain the following.

2) $(\exists v_1,...,v_{3r} \in E_{j+1})(v_1,...,v_{3r} \leq \text{CODE}(c_{n+1};x_1,...,x_r) \wedge v_{r+1},...,v_{2r} \leq \uparrow p(|v_{2r+1}-v_1,...,v_{3r}-v_r|) \wedge \text{INCODE}(\text{CODE}(c_{n+1};x_1,...,x_r)) = P(v_{2r+1}-v_1,...,v_{3r}-v_r) \wedge \varphi(v_{2r+1}-v_1,...,v_{3r}-v_r,v_{r+1},...,v_{2r})) \leftrightarrow a\text{CODE}(c_{n+1};x_1,...,x_r)+b \notin E_{j+1} \leftrightarrow d\text{CODE}(c_{n+1};x_1,...,x_r)+e \in E_{j+1}$.

By Lemma 5.3.8, $8P(x_1,...,x_r)+1 < \log(c_{n+1})$. Using Lemma 5.3.11,

3) $(\exists v_1,...,v_{3r} \in E_{j+1})(v_1,...,v_{2r} \leq \text{CODE}(c_{n+1};x_1,...,x_r) \wedge v_{r+1},...,v_{2r} \leq \uparrow p(|v_{2r+1}-v_1,...,v_{3r}-v_r|) \wedge P(x_1,...,x_r) = P(v_{2r+1}-$

$v_1,\ldots,v_{3r}-v_r) \wedge \varphi(v_{2r+1}-v_1,\ldots,v_{3r}-v_r,v_{r+1},\ldots,v_{2r})) \leftrightarrow$
$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1} \leftrightarrow d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}$.

4) $(\exists v_1,\ldots,v_{3r} \in E_{j+1})(v_1,\ldots,v_{2r} \leq \mathrm{CODE}(c_{n+1};x_1,\ldots,x_r) \wedge v_{r+1},\ldots,v_{2r} \leq \uparrow p(|v_{2r+1}-v_1,\ldots,v_{3r}-v_r|) \wedge x_1 = v_{2r+1}-v_1 \wedge \ldots \wedge x_r = v_{3r}-v_r \wedge \varphi(v_{2r+1}-v_1,\ldots,v_{3r}-v_r,v_{r+1},\ldots,v_{2r})) \leftrightarrow$
$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1} \leftrightarrow d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}$.

By Lemma 5.3.12,

5) $(\exists v_{r+1},\ldots,v_{2r} \in E_{j+1})(v_{r+1},\ldots,v_{2r} \leq \mathrm{CODE}(c_{n+1};x_1,\ldots,x_r) \wedge v_{r+1},\ldots,v_{2r} \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,v_{r+1},\ldots,v_{2r})) \leftrightarrow$
$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1} \leftrightarrow d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}$.

Note that the application of Lemma 5.3.12 to the $x = 1$ requires $1 = 4-3$, and $4 \leq \uparrow 1(1)$ is false. However, $4 \leq \uparrow 2(1)$ is true. This explains why we require $p \geq 2$.

By $x_1,\ldots,x_r \leq c_n$ and Lemma 5.3.8,

6) $(\exists v_{r+1},\ldots,v_{2r} \in E_{j+1})(v_{r+1},\ldots,v_{2r} \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,v_{r+1},\ldots,v_{2r})) \leftrightarrow a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1} \leftrightarrow d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}$.

QED

LEMMA 5.3.14. Let $r \geq 1$, $i_1,\ldots,i_r \geq 1$, and $\varphi(x_1,\ldots,x_{2r})$ be a quantifier free formula of L. Suppose $(\forall v_1,\ldots,v_r \in E_2)(\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r))$. Then for all $j \geq 1$, $(\forall v_1,\ldots,v_r \in E_j)(\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r))$.

Proof: Let $r,\varphi,i_1,\ldots,i_r$ be as given. Fix $n > i_1,\ldots,i_r$.

We apply Lemma 5.3.13. Let $a,b,d,e \in N\backslash\{0\}$ be such that the following holds. For all $j \geq 1$,

$$(\exists v_1,\ldots,v_{r+1} \in E_{j+1})(v_1,\ldots,v_r \leq |c_{i\_1},\ldots,c_{i\_r},c_n|\uparrow\uparrow \wedge v_1,\ldots,v_r \leq c_n \wedge \neg\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r)) \leftrightarrow a\mathrm{CODE}(c_{n+1};c_{i\_1},\ldots,c_{i\_r},c_n)+b \notin E_{j+1} \leftrightarrow d\mathrm{CODE}(c_{n+1};c_{i\_1},\ldots,c_{i\_r},c_n)+e \in E_{j+1}.$$

$$(\exists v_1,\ldots,v_r \in E_{j+1})(v_1,\ldots,v_r \leq c_n \wedge \neg\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r)) \leftrightarrow a\mathrm{CODE}(c_{n+1};c_{i\_1},\ldots,c_{i\_r},c_n)+b \notin E_{j+1} \leftrightarrow d\mathrm{CODE}(c_{n+1};c_{i\_1},\ldots,c_{i\_r},c_n)+e \in E_{j+1}.$$

By hypothesis,

$$\neg(\exists v_1,\ldots,v_r \in E_2)$$
$$(v_1,\ldots,v_r \le c_n \wedge \neg\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r)).$$

$$aCODE(c_{n+1};c_{i\_1},\ldots,c_{i\_r},c_n)+b \in E_2.$$

Now let $j \ge 1$. Then

$$aCODE(c_{n+1};c_{i\_1},\ldots,c_{i\_r},c_n)+b \in E_{j+1}.$$

Hence

$$\neg(\exists v_1,\ldots,v_r \in E_{j+1})$$
$$(v_1,\ldots,v_r \le c_n \wedge \neg\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r)).$$

I.e.,

$$(\forall v_1,\ldots,v_r \in E_{j+1})$$
$$(v_1,\ldots,v_r \le c_n \rightarrow \varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r)).$$

Since $n \ge i_1,\ldots,i_r$ is arbitrary and the c's have no upper bound in A, we have

$$(\forall v_1,\ldots,v_r \in E_{j+1})(\varphi(c_{i\_1},\ldots,c_{i\_r},v_1,\ldots,v_r)).$$

QED

DEFINITION 5.3.12. We say that $t(v_1,\ldots,v_r)$ is a $(p,<\infty)$ term of L if and only if

i. $t(v_1,\ldots,v_r)$ is a term of L.
ii. p is a positive standard integer.
iii. There exists a standard integer q such that for all $x_1,\ldots,x_r \in A$, $p|x_1,\ldots,x_r| \le t(x_1,\ldots,x_r) \le q|x_1,\ldots,x_r|$.

LEMMA 5.3.15. Let $r \ge 1$, $t(v_1,\ldots,v_{2r})$ be a $(1,<\infty)$ term of L, and $i_1,\ldots,i_{2r} \ge 1$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min. For all $j \ge 1$, $y_1,\ldots,y_r \in E_j$, $y_1,\ldots,y_r \le \min(c_{i\_1},\ldots,c_{i\_r})$,

$$t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \in E_{j+1} \leftrightarrow$$
$$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in E_{j+1}.$$

Proof: By Lemma 5.3.3 viii), this holds for $j = 2$. In fact, in the case $j = 2$, we have the equivalence for any term t.

We will use Lemma 5.3.14 to argue that it holds for any $j \ge 1$. But there are many details that need to be checked.

By Lemma 5.3.9, let $a,b \in N\backslash\{0\}$ be such that the following holds. For all $j \geq 1$ and $x \in \alpha(E_j;1,<\infty)$,

$$1)\ x \in E_{j+1} \leftrightarrow ax+b \notin E_{j+1}.$$

Let $r,t,i_1,\ldots,i_{2r}$ be as given. Fix $n = \min(i_1,\ldots,i_{2r})$.

Since t is a $(1,<\infty)$ term, for all $j \geq 1$ and $y_1,\ldots,y_r \in E_j$,

$$2)\ t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r),\ t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in \alpha(E_j;1,<\infty).$$

By 1),2), for all $j \geq 1$ and $y_1,\ldots,y_r \in E_j$,

$$3)\ t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \in E_{j+1} \rightarrow at(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \notin E_{j+1}$$
$$\wedge$$
$$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in E_{j+1} \rightarrow at(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \notin E_{j+1}.$$

By Lemma 5.3.3 viii), for all $y_1,\ldots,y_r \in \alpha(E_2) \cap [0,c_n]$,

$$4)\ t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \in E_3 \rightarrow t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in E_3$$
$$\wedge$$
$$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in E_3 \rightarrow t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \in E_3.$$

By 1),2),4), for all $y_1,\ldots,y_r \in E_2 \cap [0,c_n]$,

$$5)\ t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \in E_3 \rightarrow at(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \notin E_3$$
$$\wedge$$
$$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in E_3 \rightarrow at(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \notin E_3.$$

By elementary logical manipulations from 5, for all $y_1,\ldots,y_r \in E_2 \cap [0,c_n]$,

$$t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \notin E_3 \vee at(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \notin E_3$$
$$\wedge$$
$$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \notin E_3 \vee at(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \notin E_3.$$

$$(\forall u,v \in E_3)(t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \neq u \vee at(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \neq v)$$

$$\wedge$$

$$(\forall u,v \in E_3)(t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \neq u \vee at(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \neq v).$$

$$6)\ (\forall u_1,u_2,u_3,u_4 \in E_3)$$
$$(t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \neq u_1 \vee at(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \neq u_2$$
$$\wedge$$
$$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \neq u_3 \vee at(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \neq u_4).$$

Write 6) in the form

$$7)\ (\forall v_{3r+1},v_{3r+2},v_{3r+3},v_{3r+4} \in E_3)$$
$$\psi(c_{i\_1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r+4}).$$

where ψ is given by

$$8)\ \psi(v_1,\ldots,v_{3r+4}) =$$
$$(t(v_1,\ldots,v_r,v_{2r+1},\ldots,y_{3r}) \neq v_{3r+1} \vee at(v_{r+1},\ldots,v_{2r},v_{2r+1},\ldots,v_{3r})+b \neq v_{3r+2}$$
$$\wedge$$
$$t(v_{r+1},\ldots,v_{2r},\ v_{2r+1},\ldots,y_{3r}) \neq v_{3r+3} \vee at(v_1,\ldots,v_r,v_{2r+1},\ldots,y_{3r})+b \neq v_{3r+4}).$$

To recapitulate, we have

$$9)\ (\forall v_{2r+1},\ldots,v_{3r} \in E_2)(v_{2r+1},\ldots,v_{3r} \leq c_n \rightarrow$$
$$(\forall v_{3r+1},v_{3r+2},v_{3r+3},v_{3r+4} \in E_3)$$
$$(\psi(c_{i\_1},\ldots,c_{i\_2r},v_{2r+1},\ldots,y_{3r+4}))).$$

We can weaken 9) to

$$10)\ (\forall v_{2r+1},\ldots,y_{3r+4} \in E_2)$$
$$(v_{2r+1},\ldots,y_{3r} \leq c_n \rightarrow \psi(c_{i\_1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r+4})).$$

We now want to apply Lemma 5.3.14 to obtain 10) with E_2 replaced by any E_j, $j \geq 1$. Lemma 5.3.14 requires that we quantify over $v_1,\ldots,v_{r'}$, and use r' constants, where $r' \geq 1$. We can set $r' = 3r+4$ and add $2r+3$ dummy constants.

Hence for all $j \geq 1$,

$$11)\ (\forall v_{2r+1},\ldots,y_{3r+4} \in E_{j+1})$$
$$(v_{2r+1},\ldots,y_{3r} \leq c_n \rightarrow \psi(c_{i\_1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r+4})).$$

We can now perform the above rewriting in reverse. By 11), for all $j \geq 1$ and $v_{2r+1},\ldots,y_{3r} \in E_j \cap [0,c_n]$,

$$12)\ (\forall v_{3r+1},v_{3r+2},v_{3r+3},v_{3r+4} \in E_{j+1})(\psi(c_{i\_1},\ldots,c_{i\_2r},v_{2r+1},\ldots,y_{3r+4})).$$

By 8),12), for all $j \geq 1$ and $v_{2r+1},\ldots,y_{3r} \in E_j \cap [0,c_n]$,

$$13)\ (\forall v_{3r+1},v_{3r+2},v_{3r+3},v_{3r+4} \in E_{j+1}) (t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \neq v_{3r+1} \vee at(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r})+b \neq v_{3r+2} \wedge t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \neq v_{3r+3} \vee at(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r})+b \neq v_{3r+4}).$$

By logical manipulations, for all $j \geq 1$ and $v_{2r+1},\ldots,v_{3r} \in E_j \cap [0,c_n]$,

$$14)\ (\forall v_{3r+1},v_{3r+2} \in E_{j+1})(t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \neq v_{3r+1} \vee at(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r)+b \neq v_{3r+2}) \wedge (\forall v_{3r+3},v_{3r+4} \in E_{j+1})(t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \neq v_{3r+3} \vee at(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r)+b \neq v_{3r+4}).$$

$$15)\ t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \notin E_{j+1} \vee at(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r})+b \notin E_{j+1} \wedge t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \notin E_{j+1} \vee at(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r})+b \notin E_{j+1}.$$

$$16)\ t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1} \rightarrow at(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r})+b \notin E_{j+1} \wedge t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1} \rightarrow at(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r})+b \notin E_{j+1}.$$

By 1),2),16),

$$17)\ t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1} \rightarrow t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1} \wedge t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1} \rightarrow t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1}.$$

$$18)\ t(c_{i\_1},\ldots,c_{i\_r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1} \leftrightarrow t(c_{i\_r+1},\ldots,c_{i\_2r},v_{2r+1},\ldots,v_{3r}) \in E_{j+1}$$

as required. QED

LEMMA 5.3.16. Let $r \geq 1$ and $t(v_1,...,v_r)$ be a term of L. There exists a $(1,<\infty)$ term $t'(v_1,...,v_{r+1})$ such that the following holds. Let $n,j \geq 1$ and $x_1,...,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$. Then $t(x_1,...,x_r) \in E_{j+1} \leftrightarrow t'(x_1,...,x_r,c_{n+1}) \in E_{j+1}$.

Proof: Let r,t be as given. By Lemma 5.3.8, let $p \geq 2$ be such that for all $n \geq 1$, $x_1,...,x_r \leq c_n$, we have $t(x_1,...,x_r) \leq \uparrow p(c_n)$. By Lemma 5.3.13, let $a,b \in N\backslash\{0\}$ be such that the following holds. Let $j,n \geq 1$ and $x_1,...,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$. Then

$$(\exists y \in E_{j+1})(y \leq \uparrow p(|x_1,...,x_r|) \wedge y = t(x_1,...,x_r)) \leftrightarrow \text{aCODE}(c_{n+1};x_1,...,x_r)+b \in E_{j+1}.$$

By the choice of p and $|x_1,...,x_r| \leq c_n$, we have

$$t(x_1,...,x_r) \in E_{j+1} \leftrightarrow \text{aCODE}(c_{n+1};x_1,...,x_r)+b \in E_{j+1}.$$

So we can set $t'(v_1,...,v_{r+1}) = \text{aCODE}(v_{r+1};v_1,...,v_r)+b$ if $|v_1,...,v_{r+1}| \leq \text{aCODE}(v_{r+1};v_1,...,v_r)+b \leq 16a|v_1,...,v_{r+1}|$; $|v_1,...,v_{r+1}|$ otherwise. Obviously t' is a $(1,<\infty)$ term, and for all $x_1,...,x_r \leq c_n$, $t'(x_1,...,x_r,c_{n+1}) = \text{aCODE}(c_{n+1};x_1,...,x_r)+b \in [c_{n+1},16ac_{n+1}]$. QED

LEMMA 5.3.17. Let $r,j \geq 1$ and $t(v_1,...,v_{2r})$ be a term of L. Let $i_1,...,i_{2r} \geq 1$, and $y_1,...,y_r \in E_j$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(c_{i\_1},...,c_{i\_r})$. Then

$$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E_{j+1} \leftrightarrow t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E_{j+1}.$$

Proof: Let $r,t(v_1,...,v_{2r})$ be as given. Let $t'(x_1,...,x_{2r+1})$ be as given by Lemma 5.3.16. Let $j,i_1,...,i_{2r},y_1,...,y_r$ be as given. Let $n > \max(i_1,...,i_{2r})$. Obviously, $y_1,...,y_r \leq c_n$. By Lemma 5.3.16,

$$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E_{j+1} \leftrightarrow t'(c_{i\_1},...,c_{i\_r},y_1,...,y_r,c_{n+1}) \in E_{j+1}.$$

$$t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E_{j+1} \leftrightarrow t'(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r,c_{n+1}) \in E_{j+1}.$$

Since t' is a $(1,<\infty)$ term, we see that by Lemma 5.3.15,

$t'(c_{i\_1},...,c_{i\_r},y_1,...,y_r,c_{n+1}) \in E_{j+1} \leftrightarrow$
$t'(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r,c_{n+1}) \in E_{j+1}$.

Hence

$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E_{j+1} \leftrightarrow$
$t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E_{j+1}$.

QED

LEMMA 5.3.18. There exists a countable structure M = $(A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...)$ such that the following holds.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0_1,L)$;
ii) $E \subseteq A\backslash\{0\}$;
iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements in $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;
iv) Let $r,n \geq 1$, $t(v_1,...,v_r)$ be a term of L, and $x_1,...,x_r \leq c_n$. Then $t(x_1,...,x_r) < c_{n+1}$;
v) $2\alpha(E;1,<\infty)+1$, $3\alpha(E;1,<\infty)+1 \subseteq E$;
vi) Let $r \geq 1$, $a,b \in N$, and $\varphi(v_1,...,v_r)$ be a quantifier free formula of L. There exist $d,e,f,g \in N\backslash\{0\}$ such that for all $x_1 \in \alpha(E;1,<\infty)$, $(\exists x_2,...,x_r \in E)(x_2,...,x_r \leq ax_1+b \wedge \varphi(x_1,...,x_r)) \leftrightarrow dx_1+e \notin E \leftrightarrow fx_1+g \in E$;
vii) Let $r \geq 1$, $p \geq 2$, and $\varphi(v_1,...,v_{2r})$ be a quantifier free formula of L. There exist $a,b,d,e \in N\backslash\{0\}$ such that the following holds. Let $n \geq 1$ and $x_1,....,x_r \in \alpha(E;1,<\infty) \cap [0,c_n]$. Then
$(\exists y_1,...,y_r \in E)(y_1,...,y_r \leq \uparrow p(|x_1,...,x_r|) \wedge$
$\varphi(x_1,...,x_r,y_1,...,y_r)) \leftrightarrow$
$aCODE(c_{n+1};x_1,...,x_r)+b \notin E \leftrightarrow$
$dCODE(c_{n+1};x_1,...,x_r)+e \in E$. Here CODE is as defined just before Lemma 5.3.11;
viii) Let $k,n,m \geq 1$, and $x_1,...,x_k \leq c_n < c_m$, where $x_1,...,x_k \in \alpha(E;1,<\infty)$. Then $CODE(c_m;x_1,...,x_k) \in E$;
ix) Let $r \geq 1$ and $t(v_1,...,v_{2r})$ be a term of L. Let $i_1,...,i_{2r} \geq 1$ and $y_1,...,y_r \in E$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and min, and $y_1,...,y_r \leq \min(c_{i\_1},...,c_{i\_r})$. Then
$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E \leftrightarrow$
$t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E$.

Proof: Let this M be the same as the M given by Lemma 5.3.3, except that instead of using $E_1,E_2,...$, we use $E = \bigcup\{E_n: n \geq 1\}$. We also use the enumeration $c_1 < c_2 < ...$ of E_1.

Claim i) is the same as Lemma 5.3.3 i).

Claim ii) is immediate from Lemma 5.3.3 ii).

Claim iii) is from Lemmas 5.3.3 iii), 5.3.4, 5.3.6, and the definition of the c's.

Claim iv) is from Lemma 5.3.8.

Claim v) is immediate from Lemma 5.3.3 vi).

For claim vi), let $r,a,b,\varphi(v_1,\ldots,v_r)$ be as given. By Lemma 5.3.10, let $d,e,f,g \in N\backslash\{0\}$ be such that the following holds. Let $j \geq 1$ and $x_1 \in \alpha(E_j;1,<\infty)$. Then

$$(\exists x_2,\ldots,x_r \in E_{j+1})(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)) \leftrightarrow dx_1+e \notin E_{j+1} \leftrightarrow fx_1+g \in E_{j+1}.$$

Let $x_1 \in \alpha(E;1,<\infty)$. For all $j \geq 1$, if $x_1 \in \alpha(E_j;1,<\infty)$, then

1) $(\exists x_2,\ldots,x_r \in E_{j+1})(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)) \leftrightarrow dx_1+e \notin E_{j+1} \leftrightarrow fx_1+g \in E_{j+1}$.

We must verify that

$$(\exists x_2,\ldots,x_r \in E)(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)) \leftrightarrow dx_1+e \notin E \leftrightarrow fx_1+g \in E.$$

First assume

2) $(\exists x_2,\ldots,x_r \in E)(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r))$.

Let j be such that

3) $x_1 \in \alpha(E_j;1,<\infty)$.
$(\exists x_2,\ldots,x_r \in E_{j+1})(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r))$.

By 1),3),

$$dx_1+e \notin E_{j+1}.$$
$$fx_1+g \in E_{j+1}.$$

Since j can be raised arbitrarily,

$$dx_1+e \notin E.$$
$$fx_1+g \in E.$$

Next assume

$$4)\ dx_1+e \notin E.$$

By 1),4), for all $j \geq 1$, if $x_1 \in \alpha(E_j;1,<\infty)$ then

$$(\exists x_2,\ldots,x_r \in E_{j+1})(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)).$$
$$fx_1+g \in E_{j+1}.$$
$$(\exists x_2,\ldots,x_r \in E)(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)).$$
$$fx_1+g \in E.$$

Finally assume

$$5)\ fx_1+g \in E.$$

By 1), for all $j \geq 1$, if $fx_1+g \in E_{j+1}$ and $x_1 \in \alpha(E_j;1,<\infty)$, then

$$(\exists x_2,\ldots,x_r \in E_{j+1})(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)).$$
$$dx_1+e \notin E_{j+1}.$$

Since we can choose such a j to be arbitrarily large,

$$(\exists x_2,\ldots,x_r \in E)(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)).$$
$$dx_1+e \notin E.$$

For claim vii), let $r,p,\varphi(v_1,\ldots,v_{2r})$ be as given. By Lemma 5.3.13, let $a,b,d,e \in N\backslash\{0\}$ be such that the following holds. For all $j,n \geq 1$ and $x_1,\ldots,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$,

$$(\exists y_1,\ldots,y_r \in E_{j+1})(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge$$
$$\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow$$
$$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1} \leftrightarrow$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}.$$

Fix $n \geq 1$ and $x_1,\ldots.,x_r \in \alpha(E;1,<\infty) \cap [0,c_n]$. Then for all $j \geq 1$ and $x_1,\ldots,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$,

$$6)\ (\exists y_1,\ldots,y_r \in E_{j+1})(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge$$
$$\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r) \leftrightarrow$$
$$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1} \leftrightarrow$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}.$$

We must verify that

$$(\exists y_1,\ldots,y_r \in E)(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge$$

$$\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r)) \leftrightarrow$$
$$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E \leftrightarrow$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E.$$

First assume

$$7)\ (\exists y_1,\ldots,y_r \in E)(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,y_1,\ldots,y_r)).$$

Let j ≥ 1 be such that

$$x_1,\ldots,x_r \in \alpha(E_j;1,<\infty).$$
$$(\exists y_1,\ldots,y_r \in E_{j+1})(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,y_1,\ldots,y_r)).$$

By 6),7),

$$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E_{j+1}.$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}.$$

Since j can be raised arbitrarily,

$$a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E.$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E.$$

Now assume

$$8)\ a\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E.$$

By 6),8), for all $j \geq 1$ and $x_1,\ldots,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$,

$$(\exists y_1,\ldots,y_r \in E_{j+1})(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,y_1,\ldots,y_r)).$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}.$$
$$(\exists y_1,\ldots,y_r \in E)(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,y_1,\ldots,y_r)).$$
$$d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E.$$

Finally assume

$$9)\ d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E.$$

By 6), for all $j \geq 1$ such that $x_1,\ldots,x_r \in \alpha(E_j;1,<\infty) \cap [0,c_n]$ and $d\mathrm{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E_{j+1}$,

$$(\exists y_1,\ldots,y_r \in E_{j+1})(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge \varphi(x_1,\ldots,x_r,y_1,\ldots,y_r).$$

$$aCODE(c_{n+1};x_1,...,x_r)+b \notin E_{j+1}.$$

Since we can choose arbitrarily large such j,

$$(\exists y_1,...,y_r \in E)(y_1,...,y_r \leq \uparrow p(|x_1,...,x_r|) \wedge \varphi(x_1,...,x_r,y_1,...,y_r).$$
$$aCODE(c_{n+1};x_1,...,x_r)+b \notin E.$$

For claim viii), let $k,n,m,x_1,...,x_k$ be as given. Let $j \geq 1$ be such that $x_1,...,x_k \in \alpha(E_j;1,<\infty)$. By Lemma 5.3.11, $CODE(c_m;x_1,...,x_k) \in E_{j+1}$. Hence $CODE(c_m;x_1,...,x_k) \in E$.

For claim ix), let $r,t,i_1,...,i_{2r},y_1,...,y_r$ be as given. By Lemma 5.3.17, for all $j \geq 1$, if $y_1,...,y_r \in E_j$ then

$$10)\ t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E_{j+1} \leftrightarrow t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E_{j+1}.$$

We must verify that

$$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E \leftrightarrow t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E.$$

First assume $t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E$. Let $j \geq 1$ be such that

$$11)\ y_1,...,y_r \in E_j.$$
$$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E_{j+1}.$$

By 10),11),

$$t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E_{j+1}.$$
$$t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E.$$

Finally, assume $t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E$. Let $j \geq 1$ be such that

$$12)\ y_1,...,y_r \in E_j.$$
$$t(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r) \in E_{j+1}.$$

By 10),12),

$$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E_{j+1}.$$
$$t(c_{i\_1},...,c_{i\_r},y_1,...,y_r) \in E.$$

QED