5.5. Comprehension, indiscernibles.

We fix $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,\ldots)$ and terms $t_1,t_2,\ldots$ of $L(E)$ be given as in Lemma 5.4.17.

We now consider unbounded quantifiers. Below, Q indicates either $\forall$ or $\exists$. All formulas of $L(E)$ are interpreted in M.

LEMMA 5.5.1. Let $n,m \geq 0$, $r \geq 1$, and $\varphi(v_1,\ldots,v_{n+m})$ be a quantifier free formula of $L(E)$. Let $x_{n+1},\ldots,x_{n+m} \in E \cap [0,c_r]$. Then
$(Q_n x_n \in E)\ldots(Q_1 x_1 \in E)(\varphi(x_1,\ldots,x_{n+m})) \leftrightarrow$
$(Q_n x_n \in E \cap [0,c_{r+1}])\ldots(Q_1 x_1 \in E \cap [0,c_{r+n}])(\varphi(x_1,\ldots,x_{n+m}))$.

Proof: We prove the following statement by induction on $n \geq 0$.

Let $m \geq 0$, $r \geq 1$, $\varphi(x_1,\ldots,x_{n+m})$ be a quantifier free formula in $L(E)$, $Q_1,\ldots,Q_n$ be quantifiers, and $x_{n+1},\ldots,x_{n+m} \in E \cap [0,c_r]$. Then

$$(Q_n x_n \in E)\ldots(Q_1 x_1 \in E)(\varphi(x_1,\ldots,x_{n+m})) \leftrightarrow$$
$$(Q_n x_n \in E \cap [0,c_{r+1}])\ldots(Q_1 x_1 \in E \cap [0,c_{r+n}])(\varphi(x_1,\ldots,x_{n+m})).$$

The basis case $n = 0$ is trivial. Assume this is true for a given $n \geq 0$. Let $m \geq 0$, $r \geq 1$, and $\varphi(x_1,\ldots,x_{n+1+m})$ be a quantifier free formula in $L(E)$. Let $x_{n+2},\ldots,x_{n+1+m} \in E \cap [0,c_r]$. We wish to verify that

$$(Q_{n+1} x_{n+1} \in E)\ldots(Q_1 x_1 \in E)(\varphi(x_1,\ldots,x_{n+1+m})) \leftrightarrow$$
$$(Q_{n+1} x_{n+1} \in E \cap [0,c_{r+1}])\ldots(Q_1 x_1 \in E \cap [0,c_{r+n+1}])$$
$$(\varphi(x_1,\ldots,x_{n+1+m})).$$

By duality, we may assume that Q_{n+1} is $\exists$. Thus we wish to verify that

1) $$(\exists x_{n+1} \in E)(Q_n x_n \in E)\ldots(Q_1 x_1 \in E)(\varphi(x_1,\ldots,x_{n+1+m}))$$
$$\leftrightarrow$$
$$(\exists x_{n+1} \in E \cap [0,c_{r+1}])(Q_n x_n \in E \cap [0,c_{r+2}])\ldots$$
$$(Q_1 x_1 \in E \cap [0,c_{r+n+1}])(\varphi(x_1,\ldots,x_{n+1+m})).$$

Let $x_{n+1} \in E \cap [0,c_{r+1}]$ witness the right side of 1). I.e.,

2) $$(Q_n x_n \in E \cap [0,c_{r+2}])\ldots(Q_1 x_1 \in E \cap [0,c_{r+n+1}])(\varphi(x_1,\ldots,x_{n+1+m})).$$

According to the induction hypothesis applied to $m+1, r+1, \varphi(x_1,...,x_{n+1+m}), Q_1,...,Q_n$, and $x_{n+1},...,x_{m+1+m}, \in E \cap [0,c_{r+1}]$, we have

$$3)\ (Q_n x_n \in E)...(Q_1 x_1 \in E)(\varphi(x_1,...,x_{n+1+m})) \leftrightarrow (Q_n x_n \in E \cap [0,c_{r+2}])...(Q_1 x_1 \in E \cap [0,c_{r+n+1}])(\varphi(x_1,...,x_{n+1+m})).$$

By 2),3),

$$(Q_n x_n \in E)...(Q_1 x_1 \in E)(\varphi(x_1,...,x_{n+1+m})),$$

which is the left side of 1) instantiated with x_{n+1}.

Finally, let $x_{n+1} \in E$ witness the left side of 1). I.e.,

$$4)\ (Q_n x_n \in E)...(Q_1 x_1 \in E)(\varphi(x_1,...,x_{n+1+m})).$$

Let $x_{n+1} \leq c_s$, $s \geq r+1$. According to the induction hypothesis applied to $m+1, s, \varphi(x_1,...,x_{n+1+m}), Q_1,...,Q_n$, and $x_{n+2},...,x_{n+1+m}, \in E \cap [0,c_s]$, we have

$$5)\ (Q_n x_n \in E)...(Q_1 x_1 \in E)(\varphi(x_1,...,x_{n+1+m})) \leftrightarrow (Q_n x_n \in E \cap [0,c_{s+1}])...(Q_1 x_1 \in E \cap [0,c_{s+n}])(\varphi(x_1,...,x_{n+1+m})).$$

By 4),5),

$$(\exists x_{n+1} \in E \cap [0,c_s])(Q_n x_n \in E \cap [0,c_{s+1}])... (Q_1 x_1 \in E \cap [0,c_{s+n}])(\varphi(x_1,...,x_{n+1+m})).$$

By Lemma 5.4.17 vii), since $x_{n+2},...,x_{n+1+m} \in E \cap [0,c_r]$, we have

$$(\exists x_{n+1} \in E \cap [0,c_{r+1}])(Q_n x_n \in E \cap [0,c_{r+2}])... (Q_1 x_1 \in E \cap [0,c_{r+n+1}])(\varphi(x_1,...,x_{n+1+m}))$$

which is the right side of 1). QED

Note that Lemmas 5.4.17 and 5.5.1 concern only the E formulas of L(E). I.e., all of the quantifiers are relativized to E. This is clear for Lemma 5.4.17 by Definitions 5.4.4, 5.4.5. Lemma 5.5.1 only involves quantifier free formulas which are inside quantifiers relativized to E.

DEFINITION 5.5.1. Let $k \geq 1$ and $R \subseteq A^k$. We say that R is M,E definable if and only if $R \subseteq E^k$, and R is definable by an E formula of L(E) with parameters from E. I.e., there exists $m \geq 1$, an E formula $\varphi(x_1,\ldots,x_{k+m})$, and $x_{k+1},\ldots,x_{k+m} \in E$, such that

$$R = \{(x_1,\ldots,x_k) \in E^k: \varphi(x_1,\ldots,x_{k+m})\}.$$

Recall the definition of x-definability (Definition 5.4.6).

DEFINITION 5.5.2. We say that R is bounded if and only if there exists $x \in E$ such that $R \subseteq [0,x]^k$.

DEFINITION 5.5.3. For all $k \geq 1$, we write X_k for the set of all bounded M,E definable k-ary relations.

LEMMA 5.5.2. Let $k \geq 1$ and $R \subseteq A^k$. The following are equivalent.
i) $R \in X_k$;
ii) R is c_n-definable for some $n \geq 1$;
iii) R is x-definable for some $x \in E$.

Proof: Let k,R be as given. We have ii) → iii) → i). So we need only prove i) → ii). Let $R \in X_k$. By choosing r to be sufficiently large, we can write

$$R = \{(x_1,\ldots,x_k) \in E^k \cap [0,c_r]: \varphi(x_1,\ldots,x_{k+m})\}$$

where $\varphi(x_1,\ldots,x_{k+m})$ is an E formula of L(E), $r \geq 1$, and $x_{k+1},\ldots,x_{k+m} \in E \cap [0,c_r]$. We can assume that φ is in prenex form. By a change of bound variables, we can assume that φ is in the form

$$(Q_1x_{k+m+1} \in E)\ldots(Q_nx_{k+m+n} \in E)(\psi(x_1,\ldots,x_{k+m+n}))$$

where $\psi(x_1,\ldots,x_{k+m+n})$ is a quantifier free formula of L(E).

Let $x_1,\ldots,x_k \in E \cap [0,c_r]$. By Lemma 5.5.1,

$$\begin{gathered} R(x_1,\ldots,x_k) \leftrightarrow \varphi(x_1,\ldots,x_{k+m}) \leftrightarrow \\ (Q_1x_{k+m+1} \in E)\ldots(Q_nx_{k+m+n} \in E)(\psi(x_1,\ldots,x_{k+m+n})) \leftrightarrow \\ (Q_1x_{k+m+1} \in E \cap [0,c_{r+1}])\ldots(Q_nx_{k+m+n} \in E \cap \\ [0,c_{r+n}])(\psi(x_1,\ldots,x_{k+m+n})). \end{gathered}$$

Since $R \subseteq E^k \cap [0,c_r]^k$, this provides a c_{r+n}-definition of R. QED

Lemma 5.5.2 reveals a considerable amount of robustness.

DEFINITION 5.5.4. We say that a k-ary relation $R \subseteq E^k$ is internal (to M) if and only if R obeys any (all) of conditions i) - iii) in Lemma 5.5.2.

DEFINITION 5.5.5. Let $k \geq 1$ and $R \subseteq A^k$. We say that $y_1,\ldots,y_9$ codes R if and only if $y_1,\ldots,y_9 \in E$ and

$$R = \{(x_1,\ldots,x_k) \in E^k \cap [0,y_9]^k: \\ t_k(x_1,\ldots,x_k,y_1,\ldots,y_8) \in E\}.$$

LEMMA 5.5.3. Every internal R is coded by some $y_1,\ldots,y_9$. For $k \geq 1$, every $y_1,\ldots,y_9 \in E$ codes some unique $R \subseteq A^k$, which must be internal.

Proof: Let R be internal. Let $n \geq 1$, where R is c_n-definable. By Lemma 5.4.17 vi), write

$$R = \{(x_1,\ldots,x_k) \in E^k \cap [0,c_n]^k: \\ t_k(x_1,\ldots,x_k,y_1,\ldots,y_8) \in E\}$$

where $y_1,\ldots,y_8 \in E$. Then R is coded by $y_1,\ldots,y_8,c_n$.

Now let $k \geq 1$, and $y_1,\ldots,y_9 \in E$. Then $y_1,\ldots,y_9$ codes

$$R = \{(x_1,\ldots,x_k) \in E^k \cap [0,y_9]^k: \\ t_k(x_1,\ldots,x_k,y_1,\ldots,y_8) \in E\}.$$

R is obviously unique (given k), bounded, and M,E definable. I.e., R is internal. QED

We now work with the second order expansion M* of M, where $M^* = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,\ldots,X_1,X_2,\ldots)$. Recall the definition of X_k (Definition 5.5.3).

We use the following language L*(E) suitable for M*.

DEFINITION 5.5.6. The first order terms of L*(E) are exactly the terms of L(E). The second order variables of L*(E) are written $V^k{}_n$, k,n, ≥ 1.
The atomic formulas of L*(E) are of the form

$$t \in E$$
$$V^k{}_n(t_1,\ldots,t_k)$$
$$s = t$$
$$s < t$$

where $s,t,t_1,...,t_k$ are first order terms of L*(E) and $k,n \geq 1$. We view E as a unary predicate symbol, rather than a second order object.

DEFINITION 5.5.7. The formulas of L*(E) are inductively defined as follows.

i) every atomic formula of L*(E) is a formula of L*(E);
ii) if φ,ψ are formulas of L*(E) then $(\neg\varphi),(\varphi\wedge\psi),(\varphi\vee\psi),(\varphi\rightarrow\psi),(\varphi\leftrightarrow\psi)$ are formulas of L*(E);
iii) if φ is a formula of L*(E) and $k,n \geq 1$, then $(\forall v_n)(\varphi)$, $(\exists v_n)(\varphi)$, $(\forall V^k_n)(\varphi)$, $(\exists V^k_n)(\varphi)$ are formulas of L*(E).

As was the case with L(E), it is the E formulas of L*(E) that we focus on.

DEFINITION 5.5.8. The E formulas of L*(E) are inductively defined as follows.

1) every atomic formula of L*(E) is a formula of L*(E);
ii) if φ,ψ are formulas of L*(E) then $(\neg\varphi),(\varphi\wedge\psi),(\varphi\vee\psi),(\varphi\rightarrow\psi),(\varphi\leftrightarrow\psi)$ are formulas of L*(E);
iii) if φ is a formula of L*(E) and $k,n \geq 1$, then $(\forall v_n \in E)(\varphi)$, $(\exists v_n \in E)(\varphi)$, $(\forall V^k_n)(\varphi)$, $(\exists V^k_n)(\varphi)$ are formulas of L*(E).

DEFINITION 5.5.9. We use

$$(\forall v_n \in E)(\varphi),\ (\exists v_n \in E)(\varphi)$$

as abbreviations for

$$(\forall v_n)(v_n \in E \rightarrow \varphi),\ (\exists v_n)(v_n \in E \wedge \varphi).$$

DEFINITION 5.5.10. The intended interpretation of L*(E) is the structure M* introduced above, where the first order quantifiers range over A, and the second order quantifiers V^k_n range over X_k.

Note that in M*, if a second order object holds at any arguments, then those arguments must have the attribute E. That is, all elements of all second order objects are tuples of elements of E.

DEFINITION 5.5.11. A relation is said to be M*,E definable if and only if it is a relation on E that is M* definable by an E formula of L*(E) with second order parameters from the various X_k and first order parameters from E only.

In practice, we will allow flexibility of notation in presenting formulas of L*(E). In particular we will often drop the subscripts or superscripts on the second order variables.

We also take advantage of the added flexibility of notation that comes from sometimes treating k-ary relations as sets of k-tuples, with the $\in$ notation.

LEMMA 5.5.4. Let $k \geq 1$ and $R \subseteq A^k$. The following are equivalent.
i) $R \in X_k$;
ii) R is c_n-definable for some $n \geq 1$;
iii) R is x-definable for some $x \in E$;
iv) R is M*,E definable and bounded.

Proof: Let k,R be as given. In light of Lemma 5.5.2, we have only to verify that iv) implies i). Let

$$R = \{(x_1,\ldots,x_k) \in E^k \cap [0,c_r]^k: \varphi(x_1,\ldots,x_{k+m},R_1,\ldots,R_n) \text{ holds in } M^*\},$$

where $k,m,n,r \geq 1$, $\varphi(V_1,\ldots,v_{k+m},V_1,\ldots,V_n)$ is an E formula of L*(E) whose free variables are among the variables $v_1,\ldots,v_{k+m},V_1,\ldots,V_n$, $x_{k+1},\ldots,x_{k+m} \in E \cap [0,c_r]$, and $R_1,\ldots,R_k$ are internal.

We can remove $R_1,\ldots,R_n$ using definitions of $R_1,\ldots,R_n$ in the form given by Lemma 5.4.17 vi).

We can also remove second order quantifiers by appropriately quantifying over codes, as can be seen from Lemma 5.5.3. This involves quantifying over nine variables. Since each second order quantifier has a definite arity, k, we are only using the fixed term t_k. We then obtain a definition of R by an E formula of L(E). Hence R is internal. QED

DEFINITION 5.5.12. The bounded comprehension axioms of L*(E) consist of all E formulas of L*(E) of the form

$$x_{k+1},\ldots,x_{k+m+1} \in E \rightarrow (\exists R)(\forall x_1,\ldots,x_k \in E)$$
$$(R(x_1,\ldots,x_k) \leftrightarrow (x_1,\ldots,x_k \leq x_{k+m+1} \wedge \varphi))$$

where $k \geq 1$, $m \geq 0$, φ is an E formula of L*(E) in which R is not free, and all first order variables free in φ are among $x_1,\ldots,x_{k+m+1}$.

LEMMA 5.5.5. The bounded comprehension axioms of L*(E) hold in M*.

Proof: Let a bounded comprehension axiom of L*(E)

$$1)\ x_{k+1},\ldots,x_{k+m+1} \in E \rightarrow (\exists R)(\forall x_1,\ldots,x_k \in E)$$
$$(R(x_1,\ldots,x_k) \leftrightarrow (x_1,\ldots,x_k \leq x_{k+m+1} \wedge \varphi))$$

be given, subject to the required syntactic conditions above. Write $\varphi = \varphi(x_1,\ldots,x_{k+m+1},V_1,\ldots,V_n)$, where $V_1,\ldots,V_n$ are distinct second order variables of L*(E), and all free variables of φ are among $x_1,\ldots,x_{k+m+1},V_1,\ldots,V_n$.

Let $x_{k+1},\ldots,x_{k+m+1} \in E$ and $R_1,\ldots,R_n \in X$ have the same respective arities as $V_1,\ldots,V_n$. Set

$$R = \{(x_1,\ldots,x_k) \in E^k\colon x_1,\ldots,x_k \leq x_{k+1}$$
$$\wedge\ \varphi(x_1,\ldots,x_{k+m+1},R_1,\ldots,R_n)\}.$$

Then R is a bounded M*,E definable relation. By Lemma 5.5.4, $R \in X_k$. Therefore R witnesses the consequent of 1). QED

LEMMA 5.5.6. Let $r \geq 1$, and $\varphi(v_1,\ldots,v_{2r})$ be an E formula of L(E). Let $1 \leq i_1,\ldots,i_{2r}$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and the same min. Let $x_1,\ldots,x_r \in E$, $x_1,\ldots,x_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$. Then $\varphi(c_{i\_1},\ldots,c_{i\_r},x_1,\ldots,x_r) \leftrightarrow \varphi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r)$.

Proof: Let $r,\varphi,i_1,\ldots,i_{2r}$ be as given. Let $t = \max(i_1,\ldots,i_{2r})$. We can assume that φ is in prenex form:

$$(Q_n v_{2r+1} \in E)\ldots(Q_1 v_{2r+n} \in E)(\psi(v_1,\ldots,v_{2r+n})).$$

where ψ is a quantifier free formula of L(E). By Lemma 5.5.1, for all $v_1,\ldots,v_{2r} \in E \cap [0,c_t]$,

$$(Q_n v_{2r+1} \in E)\ldots(Q_1 v_{2r+n} \in E)(\psi(v_1,\ldots,v_{2r+n}))$$
$$\leftrightarrow$$
$$(Q_n v_{2r+1} \in E \cap [0,c_{t+1}])\ldots(Q_1 v_{2r+n} \in E \cap$$
$$[0,c_{t+n}])(\psi(v_1,\ldots,v_{2r+n})).$$

In particular, for all $x_1,\ldots,x_r \in E$, $x_1,\ldots,x_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$,

$$1)\ (Q_n v_{2r+1} \in E)\ldots(Q_1 v_{2r+n} \in E)(\psi(c_{i\_1},\ldots,c_{i\_r},x_1,\ldots,x_r,v_{2r+1},\ldots,v_{2r+n})) \leftrightarrow (Q_n v_{2r+1} \in E \cap [0,c_{t+1}])\ldots(Q_1 v_{2r+n} \in E \cap [0,c_{t+n}])(\psi(c_{i\_1},\ldots,c_{i\_r},x1,\ldots,xr,v_{2r+1},\ldots,v_{2r+n})).$$

$$2)\ (Q_n v_{2r+1} \in E)\ldots(Q_1 v_{2r+n} \in E)(\psi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r,v_{2r+1},\ldots,v_{2r+n})) \leftrightarrow (Q_n v_{2r+1} \in E \cap [0,c_{t+1}])\ldots(Q_1 v_{2r+n} \in E \cap [0,c_{t+n}])(\psi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r,v_{2r+1},\ldots,v_{2r+n})).$$

Hence

$$3)\ \varphi(c_{i\_1},\ldots,c_{i\_r},x_1,\ldots,x_r) \leftrightarrow (Q_n v_{2r+1} \in E \cap [0,c_{t+1}])\ldots(Q_1 v_{2r+n} \in E \cap [0,c_{t+n}])(\psi(c_{i\_1},\ldots,c_{i\_r},x_1,\ldots,x_r,v_{2r+1},\ldots,v_{2r+n}))^{c\_t+n}.$$

$$4)\ \varphi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r) \leftrightarrow (Q_n v_{2r+1} \in E \cap [0,c_{t+1}])\ldots(Q_1 v_{2r+n} \in E \cap [0,c_{t+n}])(\psi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r,v_{2r+1},\ldots,v_{2r+n}))^{c\_t+n}.$$

The right sides of 3),4) are $\psi_{c\_t+n},\rho_{c\_t+n}$, respectively, where ρ,ψ begin with the quantifier Q_n. ρ,ψ are first expanded out to formulas of L(E) in the obvious way. Then the displayed quantifiers are relativized to $E \cap [0,c_{t+n}]$.

By Lemma 5.4.17 vii), for all $x_1,\ldots,x_r \in E$, $x_1,\ldots,x_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$,

$$\varphi(c_{i\_1},\ldots,c_{i\_r},x_1,\ldots,x_r) \leftrightarrow \varphi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r)$$

QED

LEMMA 5.5.7. Let $r \geq 1$, and $\varphi(v_1,\ldots,v_{2r})$ be an E formula of L*(E), with no free second order variables. Let $1 \leq i_1,\ldots,i_{2r}$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and the same min. Let $x_1,\ldots,x_r \in E$, $x_1,\ldots,x_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$. Then $\varphi(c_{i\_1},\ldots,c_{i\_r},x_1,\ldots,x_r) \leftrightarrow \varphi(c_{i\_r+1},\ldots,c_{i\_2r},x_1,\ldots,x_r)$.

Proof: By the same argument that we used in the proof of Lemma 5.5.4, using codes, we can remove all second order quantifiers in φ, thereby reducing φ to an equivalent E formula $\psi(v_1,...,v_{2r})$ of L(E). No new parameters are introduced in this process. Then apply Lemma 5.5.6. QED

LEMMA 5.5.8. There exists a countable second order structure $M^* = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...,X_1,X_2,...)$, where for all $i \geq 1$, X_i is the set of all i-ary relations on A that are c_n-definable for some $n \geq 1$; and terms $t_1,t_2,...$ of L, where for all i, t_i has variables among $x_1,...,x_{i+8}$, such that the following holds.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0_1,L)$;
ii) $E \subseteq A\backslash\{0\}$;
iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements of $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;
iv) For all $r,n \geq 1$, $\uparrow r(c_n) < c_{n+1}$;
v) $2\alpha(E;1,<\infty)+1$, $3\alpha(E;1,<\infty)+1 \subseteq E$;
vi) Let $k,n \geq 1$ and R be a c_n-definable k-ary relation. There exist $y_1,...,y_8 \in E \cap [0,c_{n+1}]$ such that $R = \{(x_1,...,x_k) \in E^k \cap [0,c_n]^k\colon t_k(x_1,...,x_k,y_1,...,y_8) \in E\}$;
vii) Let $k \geq 1$, $m \geq 0$, and φ be an E formula of L*(E) in which R is not free, where all first order variables free in φ are among $x_1,...,x_{k+m+1}$. Then $x_{k+1},...,x_{k+m+1} \in E \rightarrow (\exists R)(\forall x_1,...,x_k \in E)(R(x_1,...,x_k) \leftrightarrow (x_1,...,x_k \leq x_{k+m+1} \wedge \varphi))$;
viii) Let $r \geq 1$, and $\varphi(x_1,...,x_{2r})$ be an E formula of L*(E) with no free second order variables. Let $1 \leq i_1,...,i_{2r}$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and the same min. Let $x_1,...,x_r \in E$, $x_1,...,x_r \leq \min(c_{i\_1},...,c_{i\_r})$. Then $\varphi(c_{i\_1},...,c_{i\_r},x_1,...,x_r) \leftrightarrow \varphi(c_{i\_r+1},...,c_{i\_2r},x_1,...,x_r)$.

Proof: i),ii),iii),v),vi) are identical to i),ii),iii),v),vi) of Lemma 5.4.17. iv) follows immediately from iv) of Lemma 5.4.17. vii) is from Lemma 5.5.5. viii) is from Lemma 5.5.7. QED