5.6. Π^0_1 correct internal arithmetic, simplification.

We fix $M^* = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...,X_1,X_2,...)$ from Lemma 5.5.8. Let $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...)$. We can view the main point of this section as the derivation of a suitable form of the axiom of infinity.

Note that we have yet to use that the c's lie outside $\alpha(E;2,<\infty)$, from Lemma 5.5.8 iii). In this section, we use this in an essential way. This condition is needed in order to obtain any useable form of the axiom of infinity.

The one and only use of the fact that the c's lie outside $\alpha(E;2,<\infty)$, in this Chapter, is in the proof of Lemma 5.6.6. There we use that $c_5 \notin \alpha(E;2,<\infty)$.

We first prove the existence of a least internal set I containing 1, and closed under $+ 2c_1$ (see Lemma 5.6.7 and Definition 5.6.3). We then define natural arithmetic operations on I (see Lemma 5.6.10), resulting in the structure M(I) which satisfies PA(L) (see Lemma 5.6.11). Then we define a natural external isomorphism h from M(I) into M. We then show that M(I) satisfies PA(L) + $TR(\Pi^0_1,L)$ using the solution to Hilbert's 10th Problem (see Lemma 5.6.14).

At this point, we only care that M(I) satisfies $TR(\Pi^0_1,L)$, and that I is internally well ordered. The external h is used only to take advantage of the fact that M satisfies $TR(\Pi^0_1,L)$. We think of h as external because its range is not a subset of E.

M(I) will provide us with the arithmetic part of the structure M# in Lemma 5.6.18.

We remind the reader that for $x,y \in A$, x-y always means

$$x-y \text{ if } x \geq y;\ 0 \text{ if } x < y.$$

Recall that $\alpha(E)$ is the set of all values of terms in L at arguments from E (Definition 5.3.3).

LEMMA 5.6.1. $\alpha(E) = E-E$.

Proof: According to Lemma 5.4.12, $\alpha(E) = E-E$ holds in the structure M given by Lemma 5.3.18, which is the same as the structure M given by Lemma 5.4.17. Therefore $\alpha(E) = E-E$ also holds in the structure M^* given by Lemma 5.5.8, which is an expansion of M. QED

DEFINITION 5.6.1. We say that x is critical if and only if $x \in E-E \wedge 2xc_1+1 \in E$.

LEMMA 5.6.2. Let $p,x > 0$, where $p \in N$ and x is critical. Then $p,2x+p$ are critical.

Proof: Let p,x be as given. By Lemma 5.6.1, $p,2x+p \in E-E$. Note that $pc_1 \in \alpha(E;1,<\infty)$. Hence by Lemma 5.5.8 v), $2pc_1+1 \in 2\alpha(E;1,<\infty)+1 \subseteq E$. Hence p is critical.

By $2xc_1+1 \in E$ and Lemma 5.5.8 v),

$$|2xc_1+1,c_1| \leq (2xc_1+1)+(pc_1-1) \leq 4p|2xc_1,c_1|.$$
$$(2xc_1+1)+(pc_1-1) = 2xc_1+pc_1 = (2x+p)c_1 \in \alpha(E;1,<\infty).$$
$$2(2x+p)c_1+1 \in 2\alpha(E;1,<\infty)+1 \subseteq E.$$

Hence $2x+p$ is critical. QED

LEMMA 5.6.3. Let $x \geq 1$ be critical. Suppose that for all critical $y \in [2,x]$, there is a critical z such that $y \in \{2z,2z+1\}$. Then $x+1$ is critical and there is a critical z such that $x+1 \in \{2z,2z+1\}$.

Proof: Let $x \geq 1$, and assume the hypothesis. If $x = 1$ then by Lemma 5.6.2, $x+1 = 2$ is critical and $z = 1$ is critical and $x+1 \in \{2z,2z+1\}$. So we can assume $x \geq 2$. Hence $x \in [2,x]$. By hypothesis, let z be critical and $x \in \{2z,2z+1\}$. Then $z \geq 1$ and $z < x$.

Suppose $z = 1$. Then $x \in \{2,3\}$, and so $x+1$ is critical by Lemma 5.6.2. If $x = 2$ then $x+1 \in \{2(1),2(1)+1\}$ and 1 is critical. If $x = 3$ then $x+1 \in \{2(2),2(2)+1\}$ and 2 is critical.

We may suppose $z \geq 2$. By hypothesis, let w be critical, where $z \in \{2w,2w+1\}$. Then $w \geq 1$, and

$$x \in \{4w,4w+1,4w+2,4w+3\}.$$
$$(x+1)c_1 \in \{4wc_1+c_1,4wc_1+2c_1,4wc_1+3c_1,4wc_1+4c_1\}.$$

$$(x+1)c_1 \in \{2(2wc_1+1)+c_1-2,2(2wc_1+1)+2c_1-2,$$
$$2(2wc_1+1)+3c_1-2,2(2wc_1+1)+4c_1-2\}$$

Now each of these four terms lies in $[2wc_1+1,4(2wc_1+1)]$, and $2wc_1+1 \in E$ (since w is critical). Therefore all four terms lie in $\alpha(E;1,<\infty)$. Hence $(x+1)c_1 \in \alpha(E;1,<\infty)$. So by Lemma 5.5.8 v), $2(x+1)c_1+1 \in 2\alpha(E;1,<\infty)+1 \subseteq E$.

Since x is critical, $x \in E-E$. By Lemma 5.6.1, $x+1 \in E-E$. Hence x+1 is critical.

Note that $z+1 \in \{2w+1,2w+2\}$. Since w is critical, by Lemma 5.6.2, z+1 is critical.

Using $z \in \{2w,2w+1\}$, $x \in \{4w,4w+1,4w+2,4w+3\}$, we see that $x+1 \in \{2z+1,2(z+1)\} = \{4w+1,4w+2,4w+3,4w+4\}$. We have that z,z+1 are critical.

case 1. $x+1 = 2z+1$. Then there is a critical u such that $x+1 \in \{2u,2u+1\}$, by taking $u = z$.

case 2. $x+1 = 2(z+1)$. Then there is a critical u such that $x+1 \in \{2u,2u+1\}$, by taking $u = z+1$.

QED

DEFINITION 5.6.2. Let C be the set of all $2xc_1+1$ such that

i) x is critical $\wedge$ $x \geq 1$;
ii) for all critical $y \in [2,x]$, there exists critical z such that $y \in \{2z,2z+1\}$.

LEMMA 5.6.4. $\min(C) = 2c_1+1$. $C \subseteq E \cap \alpha(E;2,<\infty)$. $(\forall u \in C)(u+2c_1 \in C)$. C is M,E definable, with only the parameter c_1.

Proof: 1 is critical by Lemma 5.6.2. Hence $2c_1+1 \in C$, and $2c_1+1$ is the least element of C.

For the second claim, let $y \in C$, and write $y = 2xc_1+1$, x critical, $x \geq 1$. Hence $y = 2xc_1+1 \in E$. Therefore, it suffices to verify that $y \in \alpha(E;2,<\infty)$.

If $x = 1$ then $y \in \alpha(E;2,<\infty)$. Assume $x \geq 2$. Therefore $x \in [2,x]$. Let $x \in \{2z,2z+1\}$, where z is critical. If $z = 1$ then again $y \in \alpha(E;2,<\infty)$.

Assume $z \geq 2$. Then $z \in [2,x]$. Let $z \in \{2w,2w+1\}$, where w is critical. Then $x \in \{4w,4w+2,4w+1,4w+3\}$. Also $2wc_1+1 \in E$. Clearly $w \geq 1$. We have

$$y = 2xc_1+1 \in \{8wc_1+1, 8wc_1+4c_1+1, 8wc_1+2c_1+1, 8wc_1+6c_1+1\}.$$

$$y \in \{4(2wc_1+1)-3, 4(2wc_1+1)+4c_1-3, 4(2wc_1+1)+2c_1-3, 4(2wc_1+1)+6c_1-3\}.$$

Therefore $y \in \alpha(E;2,<\infty)$, using $2wc_1+1 \in E$ and c_1 as the parameters, and noting that $w \geq 1$. This establishes the second claim.

For the third claim, let $u \in C$. Write $u = 2xc_1+1$. Then $x \geq 1$ is critical. Also, for all critical $y \in [2,x]$, there exists critical z such that $y \in \{2z,2z+1\}$. Hence by Lemma 5.6.3, $x+1$ is critical and there exists critical z such that $x+1 \in \{2z,2z+1\}$.

We must verify that $u+2c_1 = 2xc_1+1+2c_1 = 2(x+1)c_1+1 \in C$. We have only to verify clause ii) in the definition of C with $x+1$ instead of x. Let $y \in [2,x+1]$ be critical. If $y \leq x$ then there exists critical z such that $y = \{2z,2z+1\}$, since $2xc_1+1 \in C$. Now suppose $y = x+1$. We have already established that $x+1$ is critical and there is a critical z such that $x+1 \in \{2z,2z+1\}$. This establishes the third claim.

For the fourth claim, we must check that $C \subseteq E$ can be defined by an E formula of L(E), using only the parameter c_1. Note that $v \in C$ if and only if $v \in E$ and

$$(\exists x)(v = 2xc_1+1 \wedge x \text{ is critical} \wedge x \geq 1 \wedge (\forall y \in [2,x])(y \text{ critical} \rightarrow (\exists z)(z \text{ is critical} \wedge y \in \{2z,2z+1\}))).$$

$$(\exists x)(v = 2xc_1+1 \wedge x \text{ is critical} \wedge x \geq 1 \wedge (\forall \text{ critical } y)(y \in [2,x] \rightarrow (\exists z)(z \text{ is critical} \wedge y \in \{2z,2z+1\}))).$$

$$(\exists x \in E-E)(v = 2xc_1+1 \wedge x \geq 1 \wedge (\forall y \in E-E)(2yc_1+1 \in E \wedge y \in [2,x] \rightarrow (\exists z \in E-E)(2zc_1+1 \in E \wedge y \in \{2z,2z+1\}))).$$

$$(\exists x_1,x_2 \in E)(v = 2(x_1-x_2)c_1+1 \wedge x_1-x_2 \geq 1 \wedge (\forall y_1,y_2 \in E)(2(y_1-y_2)c_1+1 \in E \wedge y_1-y_2 \in [2,x] \rightarrow (\exists z_1,z_2 \in E)(2(z_1-z_2)c_1+1 \in E \wedge y_1-y_2 \in \{2(z_1-z_2),2(z_1-z_2)+1\}))).$$

QED

LEMMA 5.6.5. Suppose $2(E-E)c_1+1 \neg\subseteq C \cup \{1\}$. There exists an internal subset of $C \cup \{1\}$, containing 1, and closed under $+2c_1$.

Proof: Let $x \in E-E$, $2xc_1+1 \notin C \cup \{1\}$. Then $x > 1$. By Lemma 5.6.4, $C \cap [0,2xc_1+1]$ is internal, and contains $2c_1+1$.

We claim that $C \cap [0,2xc_1+1]$ is closed under $+2c_1$. To see this, let $u \in C \cap [0,2xc_1+1]$. By Lemma 5.6.4, $u+2c_1 \in C$. Write $u = 2yc_1+1$.

If $y < x$ then $u+2c_1 = 2yc_1+1+2c_1 = 2(y+1)c_1+1 \leq 2xc_1+1$.

If $y = x$ then $u = 2xc_1+1$. This contradicts $u \in C$, $2xc_1+1 \notin C$.

If $y \geq x+1$ then $u \geq 2(x+1)c_1+1 > 2xc_1+1$. This contradicts $u \leq 2xc_1+1$. This establishes the claim.

It is now clear that $(C \cap [0,2xc_1+1]) \cup \{1\}$ contains 1, and is closed under $+2c_1$, and is internal. QED

LEMMA 5.6.6. Suppose $2(E-E)c_1+1 \subseteq C \cup \{1\}$. There exists an internal subset of $C \cup \{1\}$, containing 1, and closed under $+2c_1$.

Proof: Assume $2(E-E)c_1+1 \subseteq C \cup \{1\}$.

Suppose $C \cap [0,c_5]$ has no greatest element. Note that by Lemma 5.6.4, $(C \cap [0,c_5]) \cup \{1\}$ is an internal subset of E, containing 1.

We claim that $(C \cap [0,c_5]) \cup \{1\}$ is closed under $+2c_1$. To see this, let $u \in (C \cap [0,c_5]) \cup \{1\}$. Let $u = 2zc_1+1$. Since u is not the greatest element of $(C \cap [0,c_5])$, let $2zc_1+1 < 2wc_1+1 \in (C \cap [0,c_5]) \cup \{1\}$. By Lemma 5.6.4, $2zc_1+1+2c_1 = 2(z+1)c_1+1 \in C$. Since $w \geq z+1$, we see that $2zc_1+1+2c_1 \leq 2wc_1+1$. Hence $2zc_1+1+2c_1 \in (C \cap[0,c_5]) \cup \{1\}$. This establishes the claim.

By the claim, it suffices to assume that $C \cap [0,c_5]$ has a greatest element. Let u be the greatest element of $C \cap [0,c_5]$. We will derive a contradiction.

Since C is closed under $+2c_1$, $u+2c_1 \in C$, $u+2c_1 > c_5$, $c_5-u < 2c_1$. Since $c_5-u \in E-E$, we have $v = 2(c_5-u)c_1+1 \in 2(E-E)c_1+1 \subseteq C \cup \{1\}$.

Note that $v < 2(2c_1)c_1+1 < c_2$, by Lemma 5.5.8 iv).

Consider the following true statement about v,c_1.

$$(\exists x,y \in E)(y \leq x \wedge v = 2(x-y)c_1+1).$$

By Lemma 5.5.8 iii), let $n \geq 3$ be so large that

$$(\exists x,y \in E)(y \leq x < c_n \wedge v = 2(x-y)c_1+1).$$

By Lemma 5.5.8 viii),

$$(\exists x,y \in E)(y \leq x < c_3 \wedge v = 2(x-y)c_1+1).$$

Fix $x,y \in E$, $y \leq x < c_3$, $v = 2(x-y)c_1+1$. Then $2(x-y)c_1+1 = 2(c_5-u)c_1+1$, $x-y = c_5-u$. Hence

$$c_5 = u+(x-y).$$

By Lemma 5.6.4, $u \in \alpha(E;2,<\infty)$. Since $x-y = c_5-u < 2c_1$, we have $u > c_5-2c_1 > c_4$, using Lemma 5.5.8 iv).

We claim that $c_5 \in \alpha(E;2,<\infty)$. To see this, write

$$u = t(w_1,\ldots,w_k),\ w_1,\ldots,w_k \in E,\ k \geq 1.$$
$$2|w_1,\ldots,w_k| \leq u \leq p|w_1,\ldots,w_k|,\ p \in N.$$

By Lemma 5.5.8 iv), since $u > c_4$, we have

$$x,y < c_3 < |w_1,\ldots,w_k|$$
$$|w_1,\ldots,w_k,x,y| = |w_1,\ldots,w_k|$$
$$2|w_1,\ldots,w_k,x,y| \leq u \leq u+(x-y) = t(w_1,\ldots,w_k)+(x-y)$$
$$\leq 2p|w_1,\ldots,w_k,x,y|.$$
$$c_5 \in \alpha(E;2,<\infty).$$

using the representation $c_5 = t(w_1,\ldots,w_k)+(x-y)$ in the parameters $w_1,\ldots,w_k,x,y \in E$. But this contradicts Lemma 5.5.8 iii). QED

LEMMA 5.6.7. There exists an internal subset of $C \cup \{1\}$, containing 1, and closed under $+2c_1$. $C \subseteq E \cap (2(E-E)c_1+1)$.

Proof: The first claim is by Lemmas 5.6.5 and 5.6.6. For the second claim, $C \subseteq E$ by Lemma 5.6.4. Let $u \in C$. Write $u = 2xc_1+1$, x critical. Then $x \in E-E$. Hence $u \in 2(E-E)c_1+1$. QED

DEFINITION 5.6.3. Let I be the intersection of all internal sets containing 1, and closed under $+2c_1$.

By Lemma 5.6.7, I exists.

LEMMA 5.6.8. The following hold.
i. I is the least internal set which is closed under $+2c_1$ and contains 1.
ii. $I \subseteq C \cup \{1\}$.
iii. The immediate successor in I of any $x \in I$ is $x+2c_1$.
iv. Every internal nonempty subset of I has a least element.
v. I is defined by an E formula of L*(E) with only the parameter c_1.
vi. $I \subseteq [0,c_2)$.

Proof: By Lemma 5.5.8 vii), I is an internal set. By definition, it is closed under $+2c_1$ and contains 1. Hence i) follows from the definition of I.

ii) follows from Lemma 5.6.7.

For iii), it follows from ii) that every element of I is of the form $2xc_1+1$. Let $u \in I$. Write $u = 2xc_1+1$. Now $2xc_1+1+2c_1 = 2(x+1)c_1+1 \in I$. There is no room for any element of I strictly between $2xc_1+1$ and $2(x+1)c_1+1 = u+2c_1$.

For iv), let $S \subseteq I$ be nonempty and internal. If S has no least element then let $S^* = \{x \in I: x$ is below every element of $S\}$. Obviously $S^* \subseteq I$ is a nonempty internal set containing 1 with no greatest element. Let $u \in S^*$. Let $u < v \in S^*$. Then $u+2c_1 \in I$ and $u+2c_1 \leq v$. Therefore $u+2c_1 \in S$. Thus we have shown that S^* is closed under $+2c_1$, and contains 1. Therefore $S^* = I$. This contradicts the definition of S^*.

For v), the natural formalization of the definition of I results in an E formula of L*(E) with only the parameter c_1.

For vi), by Lemma 5.5.8 iii), since the c's are unbounded in A, and I is bounded in A, let $n \geq 2$ be such that $I \subseteq [0,c_n]$. We view this inclusion as a statement about c_1,c_n. By Lemma 5.5.8 viii), the corresponding statement about c_1,c_2 holds. I.e., $I \subseteq [0,c_2]$. QED

LEMMA 5.6.9. Every element of I is of the form $2xc_1+1$, with $x \in E$-E. $x \in I \wedge x > 1 \rightarrow x-2c_1 \in I$.

Proof: For the first claim, let $u \in I$. By Lemma 5.6.8, $u \in C \cup \{1\}$. If $u = 1$ then set $x = 0$. If $u \in C$, apply the second claim of Lemma 5.6.7.

For the second claim, let $x \in I$, $x > 1$, $x-2c_1 \notin I$. Then $I \cap [0,x)$ is an internal set which contains 1.

We claim that $I \cap [0,x)$ is closed under $+2c_1$. To see this, write $x = 2c_1z+1$. Let $u = 2c_1w+1 \in I \cap [0,x)$. Then $w < z$ and $2c_1(w+1)+1 \in I$.

It remains to show that $2c_1(w+1)+1 < x$. I.e., $w+1 < z$. From $w < z$, we have $w+1 \leq z$. So we merely have to eliminate the case $w+1 = z$.

Suppose $w+1 = z$. Then $w = z-1$, $u = 2c_1(z-1)+1 = x-2c_1 \in I$. This contradicts $x-2c_1 \notin I$.

We now see that $I \cap [0,x)$ is an internal set closed under $+2c_1$, containing 1. By Lemma 5.6.8, $I \cap [0,x) = I$, contradicting $x \in I$. QED

LEMMA 5.6.10. The following hold.
i. If $2xc_1+1, 2yc_1+1 \in I$ then $2(x+y)c_1+1 \in I$.
ii. If $2xc_1+1, 2yc_1+1 \in I$ then $2xyc_1+1 \in I$.
iii. If $2xc_1+1, 2yc_1+1 \in I$ then $2(x-y)c_1+1 \in I$.
iv. If $2xc_1+1 \in I$ then $2x{\uparrow}c_1+1 \in I$.
v. If $2xc_1+1 \in I$ then $2\log(x)c_1+1 \in I$.

Proof: For i), fix $u = 2xc_1+1 \in I$. We can assume that $x > 0$. Let

$$S = \{v \in I: (\exists y)(v = 2yc_1+1 \wedge 2(x+y)c_1+1 \notin I)\} = \{v \in I: (\exists y \in E\text{-}E)(v = 2yc_1+1 \wedge 2(x+y)c_1+1 \notin I)\}.$$

This equality holds by Lemma 5.6.9.

By Lemma 5.6.8, S is internal. Assume S is nonempty. By Lemma 5.6.8, let $v = 2yc_1+1$ be the least element of S. Clearly $v > 1$, $y > 0$, and so by Lemma 5.6.9, $v-2c_1 = 2(y-1)c_1+1 \in I$. By the choice of v, $v-2c_1 \notin S$. Hence $2(x+y-1)c_1+1 \in I$. By Lemma 5.6.8, $2(x+y-1)c_1+1+2c_1 = 2c_1(x+y)+1 \in I$. This contradicts $v \in S$.

For ii), fix $u = 2xc_1+1 \in I$. We can assume that $x > 0$. Let

$$S' = \{v \in I: (\exists y)(v = 2yc_1+1 \wedge 2(xy)c_1+1 \notin I)\} = \{v \in I: (\exists y \in E-E)(v = 2yc_1+1 \wedge 2(xy)c_1+1 \notin I)\}.$$

This equality holds by Lemma 5.6.9.

By Lemma 5.6.8, S' is internal. Assume S' is nonempty. By Lemma 5.6.8, let $v = 2yc_1+1$ be the least element of S'. Clearly $v > 1$, $y > 0$, and so by Lemma 5.6.9, $v-2c_1 = 2(y-1)c_1+1 \in I$. By the choice of v, $v-2c_1 \notin S'$. Hence $2x(y-1)c_1+1 \in I$. By the first claim, since $2xc_1+1 \in I$, we have $2(x+x(y-1))c_1+1 = 2c_1(xy)+1 \in I$. This contradicts $v \in S'$.

For iii), fix $u = 2yc_1+1 \in I$, and let

$$S'' = \{v \in I: (\exists x)(v = 2xc_1+1 \wedge 2(x-y)c_1+1 \notin I)\} = \{v \in I: (\exists x \in E-E)(v = 2xc_1+1 \wedge 2(x-y)c_1+1 \notin I)\}$$

This equality holds by Lemma 5.6.9.

By Lemma 5.6.8, S'' is internal. Assume S'' is nonempty. By Lemma 5.6.8, let $v = 2xc_1+1$ be the least element of S''. Clearly $v > 1$, $x > y$, and so by Lemma 5.6.9, $v-2c_1 = 2(x-1)c_1+1 \in I$. By the choice of v, $v-2c_1 \notin S''$. Hence $2((x-1)-y)c_1+1 \in I$. Now $(x-1)-y = (x-y)-1 < x-y$. Hence $2((x-y)-1)c_1+1 \in I$. By Lemma 5.6.8, $2(x-y)c_1+1 \in I$. This contradicts $v \in S''$.

For iv), let

$$S^* = \{v \in I: (\exists x)(v = 2xc_1+1 \wedge 2x{\uparrow}c_1+1 \notin I)\} = \{v \in I: (\exists x \in E-E)(v = 2xc_1+1 \wedge 2x{\uparrow}c_1+1 \notin I)\}$$

This equality holds by Lemma 5.6.9.

By Lemma 5.6.8, S^* is internal. Assume S^* is nonempty. By Lemma 5.6.8, let $v = 2xc_1+1$ be the least element of S^*. Clearly $v > 1$, $x > 0$, and so by Lemma 5.6.9, $v-2c_1 = 2(x-1)c_1+1 \in I$. By the choice of v, $v-2c_1 \notin S^*$. Hence $2(x-1){\uparrow}c_1+1 \in I$. By the first claim, $2((x-1){\uparrow}+(x-1){\uparrow})c_1+1 = 2x{\uparrow}c_1+1 \in I$. This contradicts $v \in S^*$.

For v), let

$$S^{**} = \{v \in I: (\exists x)(v = 2xc_1+1 \wedge 2\log(x)c_1+1 \notin I)\} = \{v \in I: (\exists x \in E-E)(v = 2xc_1+1 \wedge 2\log(x)c_1+1 \notin I)\}$$

This equality holds by Lemma 5.6.9.

By Lemma 5.6.8, S** is internal. Assume S** is nonempty, By Lemma 5.6.8, let $v = 2xc_1+1$ be the least element of S**. Clearly $v > 1$, $x > 0$, and so by Lemma 5.6.19, $v-2c_1 = 2(x-1)c_1+1 \in I$. By the choice of v, $v-2c_1 \notin S^{**}$. Hence $2\log(x-1)c_1+1 \in I$. Clearly $\log(x-1) \in \{\log(x)-1,\log(x)\}$. Since $2\log(x)c_1+1 \notin I$, we have $\log(x-1) = \log(x)-1$. Hence $2(\log(x)-1)c_1+1 \in I$. by Lemma 5.6.8, $2\log(x)c_1+1 \in I$. This contradicts $v \in S^{**}$. QED

We use Lemmas 5.6.9, 5.6.10 to impose an arithmetic structure on I. We define $0' = 1$, $1' = 2c_1+1$. Let $x,y \in I$, $x = 2zc_1+1$, $y = 2wc_1+1$. We define $x +' y = 2(z+w)c_1+1$, $x -' y = 2(z-w)c_1+1$, $x \bullet' y = 2zwc_1+1$, $x\uparrow' = 2z\uparrow c_1+1$, $\log'(x) = 2\log(z)c_1+1$.

DEFINITION 5.6.4. We introduce the relational structure

$$M(I) = (I,<,0',1',+',-',\bullet',\uparrow',\log').$$

It is essential to note that by Lemma 5.6.8, M(I) is internal. I.e., the domain and component relations of M(I) are internal as relations.

DEFINITION 5.6.5. Let $h:I \to E-E$ be the one-one function defined by

$$h(2c_1x+1) = x.$$

Note that h may not be internal, because, for example, its values may not all lie in E. But h is a perfectly good external isomorphism from M(I) onto the structure

$$M|rng(h) = (rng(h),<,0,1,+,-,\bullet,\uparrow,\log)$$

which is a substructure of (a reduct of) M*. Note also that M|rng(h) may not be internal, because $rng(h) \subseteq E-E$ may not be a subset of E.

Recall from section 5.1 that $TR(\Pi^0_1,L)$ is defined to be the set of all true Π^0_1 sentences in the language based on $<,0,1,+,-,\bullet,\uparrow,\log$. Here bounded quantifiers are allowed.

It is immediate that M|rng(h) satisfies the true Π^0_1 sentences of L with **no bounded quantifiers allowed**. We have to bridge this gap.

DEFINITION 5.6.6. Let PA(L) be the usual system of Peano arithmetic for the language L. Its nonlogical axioms are as follows.

1. $x+1 \neq 0$.
2. $x+1 = y+1 \rightarrow x = y$.
3. $0+1 = 1$.
4. $x+0 = x$.
5. $x+(y+1) = (x+y)+1$.
6. $x \bullet 0 = 0$.
7. $x \bullet (y+1) = x \bullet y + x$.
8. $\neg x < 0$.
9. $x < y+1 \leftrightarrow (x < y \vee x = y)$.
10. $x \leq y \rightarrow x < y \vee x = y$.
11. $0\uparrow = 1$.
12. $(x+1)\uparrow = x\uparrow + x\uparrow$.
13. $\log(0) = 0$.
14. $y\uparrow \leq x \wedge x < (y+1)\uparrow \rightarrow \log(x) = y$.
15. $\varphi[x/0] \wedge (\forall x)(\varphi \rightarrow \varphi[x/x+1]) \rightarrow \varphi$, where φ is a formula in L.

DEFINITION 5.6.7. A strict Π^0_1 sentence is a Π^0_1 sentence without bounded quantifiers.

LEMMA 5.6.11. TR(Π^0_1,L) logically implies PA(L) without 15. M|rng(h) satisfies PA(L). M(I) satisfies PA(L).

Proof: The axioms of PA(L) without 15 are clearly true strict Π^0_1 sentences, and so by Lemma 5.5.8 i), they hold in M. Hence they also hold in the substructure M|rng(h) of M. By the external isomorphism h, they hold in M(I).

For 15, first note that by Lemma 5.6.10, M(I) satisfies that every element > 0 has an immediate predecessor. Suppose that in M(I), φ defines a subset S of I containing 0′ and closed under the +1 of M(I). Suppose $S \neq I$.

Since M(I) is internal, S is internal. Hence by Lemma 5.6.8, I\S has a least element $x \in I$. Since $x > 0'$, x has an immediate predecessor $y \in I$, with $y \in S$. Hence $x \in S$, which is a contradiction. This establishes the second claim.

The third claim follows by the isomorphism h. QED

LEMMA 5.6.12. For every $\Pi^0_1(L)$ sentence φ there is a strict $\Pi^0_1(L)$ sentence ψ such that PA(L) proves $\varphi \leftrightarrow \psi$.

Proof: By a well known normal form theorem, we fix a $\Pi^0_1(L)$ formula $\rho(x,y)$ in L with the distinct free variables x,y only, such that the following holds. For all $\Pi^0_1(L)$ sentences φ, there exists $n \in N$ such that PA(L) proves

$$1)\ (\forall x)(\rho(x,n^*)) \leftrightarrow \varphi$$

where n* is 1+1...+1, with n 1's. See, e.g., [Si99], section II.2.

From the work on Hilbert's 10th problem, there exists $k \geq 1$ and two polynomials $Q_1(x_1,\ldots,x_k,y)$, $Q_2(x_1,\ldots,x_k,y)$, with nonnegative integer coefficients, such that

$$2)\ (\forall x)(\rho(x,y)) \leftrightarrow (\forall x_1,\ldots,x_k)(Q_1(x_1,\ldots,x_k,y) \neq Q_2(x_1,\ldots,x_k,y)))$$

is true for all $y \in N$. Here all variables range over nonnegative integers. This follows immediately from the sharp form of the negative solution to Hilbert's 10th problem that asserts that every recursively enumerable subset of N is Diophantine. This is due to Y. Matiyasevich, J. Robinson, M. Davis, and H. Putnam. See, e.g., [Da73], [Mat93].

Moreover, it is well known that for a given $\rho(x,y)$, polynomials Q_1,Q_2 can be found such that PA proves: for all y, 2) holds. This is because the entire treatment of Hilbert's 10th problem can be carried out straightforwardly within PA(L). We fix such polynomials Q_1,Q_2.

(In fact, this treatment can be carried out in the very weak fragment of PA called EFA = exponential function arithmetic, which is $I\Sigma_0(\exp)$. See [HP93], p. 37, and [GD82].)

Now let φ be a $\Pi^0_1(L)$ sentence. Fix n such that 1) is provable in PA(L). Set y = n* in 2). Then PA(L) proves

$$3)\ \varphi \leftrightarrow (\forall x)(\rho(x,n^*)) \leftrightarrow (\forall x_1,\ldots,x_k)(Q_1(x_1,\ldots,x_k,n^*) \neq Q_2(x_1,\ldots,x_k,n^*))$$

and so we set

$$\psi = (\forall x_1,\ldots,x_k)(Q_1(x_1,\ldots,x_k,n^*) \neq Q_2(x_1,\ldots,x_k,n^*)).$$

QED

LEMMA 5.6.13. PA(L) + strict TR(Π^0_1,L) logically implies TR(Π^0_1,L). M|rng(h) and M(I) satisfy PA(L) + TR(Π^0_1,L).

Proof: For the first claim, let $\varphi \in$ TR(Π^0_1,L). By Lemma 5.6.12, let ψ be strict TR(Π^0_1,L), where PA(L) proves $\psi \rightarrow \varphi$. Then PA(L) + strict TR(Π^0_1,L) proves φ. Hence PA(L) + strict TR(Π^0_1,L) proves TR(Π^0_1,L).

For the second claim, by Lemma 5.6.11, M|rng(h) and M(I) satisfy PA(L). Now obviously M|rng(h) satisfies strict TR(Π^0_1,L)) since M does (Lemma 5.5.8 i)), and M|rng(h) is a substructure of M. Hence M(I) also satisfies strict TR(Π^0_1,L). Hence by the first claim, M|rng(h) and M(I) satisfy TR(Π^0_1,L). QED

Note that the definitions of CODE and INCODE from section 5.3, apply without modification to the present context.

LEMMA 5.6.14. Let k,n,m $\geq$ 1, and $x_1,\ldots,x_k \leq c_n < c_m$, where $x_1,\ldots,x_k \in E$. Then CODE($c_m;x_1,\ldots,x_k$) $\in$ E, and INCODE(CODE($c_m;x_1,\ldots,x_k$)) = P($x_1,\ldots,x_k$).

Proof: We essentially repeat the proof of Lemma 5.3.11, slightly adapted to the present context.

Let k,n,m,$x_1,\ldots,x_k$ be as given. Note that

$$(c_m \div 2)+1 \leq (\log(c_m))\uparrow \leq c_m.$$
$$2c_m \leq 4(\log(c_m))\uparrow+P(x_1,\ldots x_k)) \leq 5c_m.$$

$$4((\log(c_m))\uparrow+P(x_1,\ldots x_k)) \in \alpha(E;2,<\infty).$$
$$\mathrm{CODE}(c_m;x_1,\ldots,x_k) \in 2\alpha(E;2,<\infty)+1.$$

Hence CODE($c_m;x_1,\ldots,x_k$) $\in$ E by Lemma 5.5.8 v).

We claim that

$$1)\ \log(\mathrm{CODE}(c_m;x_1,\ldots,x_k)) = \log(c_m)+3.$$

To see this, note that

$$\log(\mathrm{CODE}(c_m;x_1,\ldots,x_k)) =$$
$$\log(8((\log(c_m))\uparrow + P(x_1,\ldots,x_k))+1) =$$
$$\log(8(\log(c_m))\uparrow + 8P(x_1,\ldots,x_k) + 1) =$$
$$\log((\log(c_m)+3)\uparrow + 8P(x_1,\ldots,x_k) + 1) \leq$$
$$\log((\log(c_m)+3)\uparrow + \log(c_m)) = \log(c_m)+3 \leq$$
$$\log((\log(c_m)+3)\uparrow + 8P(x_1,\ldots,x_k) + 1).$$

Using 1),

$$\mathrm{INCODE}(\mathrm{CODE}(c_m;x_1,\ldots,x_k)) = z \leftrightarrow$$
$$8z \leq \mathrm{CODE}(c_m;x_1,\ldots,x_k)-(\log(\mathrm{CODE}(c_m;x_1,\ldots,x_k)))\uparrow-1 < 8z+8 \leftrightarrow$$
$$8z \leq \mathrm{CODE}(c_m;x_1,\ldots,x_k)-(\log(c_m)+3)\uparrow-1 < 8z+8 \leftrightarrow$$
$$8z \leq \mathrm{CODE}(c_m;x_1,\ldots,x_k)-8((\log(c_m))\uparrow)-1 < 8z+8 \leftrightarrow$$
$$8z \leq 8P(x_1,\ldots,x_k) < 8z+8.$$

Hence

$$\mathrm{INCODE}(\mathrm{CODE}(c_m;x_1,\ldots,x_k)) = P(x_1,\ldots,x_k).$$

QED

The following will be used to give an interpretation of the $\in$ relation in the set theory K(Π) introduced below.

LEMMA 5.6.15. There is an E formula $\sigma(x_1,x_2)$ of L(E) such that the following holds. Let S be an internal set. There exist arbitrarily large $y \in E$ such that $S = \{x \in E: \sigma(x,y)\}$.

Proof: Let $S \subseteq E$ be internal. Let $n \geq 1$ be such that S is c_n-definable (see Lemma 5.5.4). By Lemma 5.5.8 vi), write

1) $S = \{x \in E \cap [0,c_n]: t_1(x,y_1,\ldots,y_8) \in E\}$

where $y_1,\ldots,y_8 \in E$. This definition of S has the parameters $c_n,y_1,\ldots,y_8$. Here t_1 is among the terms t1,t2,... given at the beginning of Lemma 5.5.8. Here t_1 is defined independently of S.

We now show that instead of using the 9 parameters $c_n,y_1,\ldots,y_8 \in E$ above, we can use a single parameter $y \in E$. In particular, we claim that there are arbitrarily large $y \in E$ such that

2) $S = \{x \in E: (\exists z_0,\ldots,z_8 \in E)(\mathrm{INCODE}(y) = P(z_0,\ldots,z_8) \wedge x \leq z_0 \wedge t_k(x,z_1,\ldots,z_8) \in E\}$.

To see this, first let $x \in S$. Then 1) holds with $c_n,y_1,...,y_8 \in E$. Set $y = CODE(c_m;c_n,y_1,...,y_8)$, where $y_1,...,y_8,c_n < c_m$. By Lemma 5.6.15, $y \in E$. Obviously $y \geq c_m$. We have

$$x \in E \cap [0,c_n] \wedge t_1(x,y_1,...,y_8) \in E.$$

Set $z_0,...,z_8 = c_n,y_1,...,y_8$, respectively. By Lemma 5.6.14,

$$INCODE(y) = INCODE(CODE(c_m;c_n,y_1,...,y_8)) = P(z_0,...,z_8).$$

Also $x \leq z_0$, $t_1(x,z_1,...,z_8) \in E$.

On the other hand, suppose

$$x \in E \wedge (\exists z_0,...,z_8 \in E)(INCODE(y) = P(z_0,...,z_8) \wedge x \leq z_0 \wedge t_1(x,z_1,...,z_8) \in E\})$$

where $y = CODE(c_m;c_n,y_1,...y_8)$.

Let $z_0,...,z_8 \in E$ be such that

$$INCODE(y) = P(z_0,...,z_8) \wedge x \leq z_0 \wedge t_1(x,z_1,...,z_8) \in E.$$

By Lemma 5.6.14, $INCODE(y) = P(c_n,y_1,...,y_8)$. Hence $c_n = z_0$, $y_1 = z_1$, ..., $y_8 = z_8$, $x \leq c_n$, and $t_1(x,y_1,...,y_8) \in E$. Hence by 1), $x \in S$.

It remains to see that S has been defined by an E formula of L(E) in x,y. It suffices to write

$$INCODE(y) = P(z_0,...,z_8)$$

as a quantifier free formula in L. This is clear from

$$INCODE(y) = u \leftrightarrow 8u \leq y-(\log(y))\uparrow-1 < 8u+8.$$

QED

We are now prepared to streamline the structure M*, retaining only what is needed to complete the construction of a model of SMAH + $TR(\Pi^0_1,L)$.

We have built quite a bit of complexity in M* in order to carry out the construction of arithmetic in M* via the internal structure M(I), and have related that arithmetic to the arithmetic of M on (a subset of) A in order to obtain Π^0_1 correctness for M(I).

Now that we have this machinery in place, we no longer need to work with any objects outside of E.

Our simplification will be formulated in terms of a first order linearly ordered set theory. We will convert M* to a model of this linearly ordered set theory whose domain is a subset of E.

We now present the language L# for linearly ordered set theory.

DEFINITION 5.6.8. The language L# is based on the following primitives.

i) variables v_n, $n \geq 1$;
ii) the constant symbols d_n, $n \geq 1$;
iii) the unary relation symbol NAT;
iv) the binary relation symbols $\in$,<;
v) the constant symbols 0,1;
vi) the unary function symbols $\uparrow$,log;
vii) the binary function symbols +,-,•;
viii) = (equality).

Note that L# includes constant symbols d_n, $n \geq 1$, whereas L, L(E), and L*(E) do not include constant symbols c_n, $n \geq 1$. The constants c_n appeared only as distinguished elements of our interpretations of the languages L, L(E), and L*(E).

DEFINITION 5.6.9. The terms of L# are built from the variables and the constant symbols of L#, using the function symbols. The atomic formulas of L# are of the form s = t, s < t, s $\in$ t, where s,t are terms of L#. Formulas of L# are defined in the usual way using the usual connectives $\neg,\wedge,\vee,\rightarrow,\leftrightarrow$, and the usual quantifiers $\forall,\exists$.

We now introduce the linearly ordered set theory K(Π) in the language L#.

DEFINITION 5.6.10. K(Π) consists of the following axioms.

1. $<$ is a linear ordering (irreflexive, transitive, connected).
2. $x \in y \rightarrow x < y$.
3. Let $1 \leq n < m$. Then $d_n < d_m$.
4. Let φ be a formula of L# in which v_1 is not free. Then $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (v_2 \leq v_3 \wedge \varphi))$.
5. Let $r \geq 1$ and $\varphi(v_1,\ldots,v_{2r})$ be a formula of L#. Let $1 \leq i_1,\ldots,i_{2r}$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min. Let $y_1,\ldots,y_r \leq \min(d_{i\_1},\ldots,d_{i\_r})$. Then $\varphi(d_{i\_1},\ldots,d_{i\_r},y_1,\ldots,y_r) \leftrightarrow \varphi(d_{i\_r+1},\ldots,d_{i\_2r},y_1,\ldots,y_r)$.
6. NAT defines a nonempty initial segment under $<$, with no greatest element, and no limit point, where all points are $< d_1$, and whose first two elements are $0 < 1$, such that $+,-,\bullet,\uparrow,\log$ map NAT into NAT.
7. $(\forall x)$(if x has an element in NAT then x has a $<$ least element).
8. Let $\varphi \in TR(\Pi^0{}_1,L)$. Take the relativization of φ to NAT.
9. $+,-,\bullet,\uparrow,\log$ have the default value 0 in case one or more arguments lie outside NAT.

DEFINITION 5.6.11. We now define the structure M# = $(D,<,\in,NAT,0,1,+,-,\bullet,\uparrow,\log,d_1,d_2,\ldots)$ as follows. Recall that we have been using the structure M* = $(A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,\ldots,X_1,X_2,\ldots)$.

By Lemma 5.6.8, $I \subseteq E \cap [0,c_2)$. Let J be the initial segment of $(E,<)$ determined by I. Take $D = E\backslash J \cup I$. I.e., D is the result of cutting down J to I in E.

Define $<$ in M# to be the restriction of $<$ in M* to D. Take $NAT(x) \leftrightarrow x \in I$.

Define $0,1,+,-,\bullet,\uparrow,\log$ of M# as follows. The 0,1 of M# are the same as the 0,1 of the structure M(I). The $+,-,\bullet,\uparrow,\log$ of M# restricted to I are the same as the $+,-,\bullet,\uparrow,\log$ of M(I). Finally, if one or more arguments lie outside I, then the $+,-,\bullet,\uparrow,\log$ of M# return the 0 of M(I).

Let $x,y \in D$. Define

$$x \in y \leftrightarrow (\sigma(x,y) \wedge x < y)$$

where σ is given by Lemma 5.6.15.

Finally, for $n \geq 1$, define $d_n = c_{n+1}$.

LEMMA 5.6.16. Let $k \geq 1$, $\varphi(v_1,...,v_k)$ be a formula of L# without any d's. There exists an E formula $\varphi'(x_1,...,x_{k+1})$ of L*(E) such that the following holds. Let $x_1,...,x_k \in D$. Then

$$\varphi(x_1,...,x_k) \text{ holds in } M\# \leftrightarrow \varphi'(x_1,...,x_k,c_1) \text{ holds in } M^*.$$

Proof: Let φ be as given. First, formally restrict the scope of all quantifiers to the formal property $x \in E \wedge (x \in I \vee (\forall v \in I)(v < x))$. The extension of this property is D.

Now replace all subformulas NAT(t) by $(\exists v)(v = t \wedge v \in I)$.

Next unravel all subformulas $s = t$, $s < t$, by using new existential quantifiers relativized to I for subterms with $0,1,+,-,\bullet,\uparrow,\log$. We can straightforwardly do this so that

i. there is at most one occurrence of a function symbol in every remaining equation.
ii. there are no occurrences of function symbols in every remaining inequality.

Now replace $v+w = z$, $v-w = z$, $v\bullet w = z$, $v\uparrow = z$, $\log(v) = z$, respectively, by

$(v \notin I \vee w \notin I \wedge z = 0) \vee (v \in I \wedge w \in I \wedge v+'w = z)$.
$(v \notin I \vee w \notin I \wedge z = 0) \vee (v \in I \wedge w \in I \wedge v-'w = z)$.
$(v \notin I \vee w \notin I \wedge z = 0) \vee (v \in I \wedge w \in I \wedge v\bullet'w = z)$.
$(v \notin I \wedge z = 0) \vee (v \in I \wedge v\uparrow' = z)$.
$(v \notin I \wedge z = 0) \vee (v \in I \wedge \log'(v) = z)$.

Then replace $v+'w = z$, $v-'w = z$, $v\bullet'w = z$, $v\uparrow' = z$, $\log'(v) = z$, respectively, by their definitions given right after the proof of Lemma 5.6.10.

Now replace 0 by 1 and 1 by $2c_1+1$.

Next replace atomic subformulas $z \in w$ by $\sigma(z,w)$, given by Definition 5.6.3.

Finally, replace all $v \in I$ by the definition of I in Definition 5.6.3.

The parameter c_1, only, appears in the definition of I, and the definitions of $1,+,-,\bullet,\uparrow,\log$. QED

LEMMA 5.6.17. $M\# = (D,<,\in,NAT,0,1,+,-,\bullet,\uparrow,\log,d_1,d_2,...)$ satisfies $K(\Pi)$, where $d_1,d_2,...$ forms a strictly increasing sequence from D without an upper bound.

Proof: Axioms 1,2,3,9 are evident by construction.

For axiom 4, let $\varphi(v_2,...,v_k)$ be as given, $k \geq 1$. Let $x_2,...,x_k \in D$. Define

$$S = \{x_2 \in D: x_2 \leq x_3 \wedge \varphi(x_2,...,x_k) \text{ holds in } M\#\}.$$

By Lemma 5.6.16, we can write S in the form

$$S = \{x_2 \in D: x_2 \leq x_3 \wedge \varphi'(x_2,...,x_k,c_1) \text{ holds in } M^*\}$$

where $\varphi'(v_2,...,v_{k+1})$ is an E formula of $L^*(E)$. Hence S is internal. By Lemma 5.6.15, let $y \in E$, $y > x_3,c_2$, be such that

$$S = \{x \in E: \sigma(x,y)\}.$$

Note that since $y \in E$ and $y > c_2$, we have $y \in D$.

Since $S \subseteq D$, $y \in D$, and S is strictly bounded above by y, we have

$$S = \{x \in D: x \in_{M\#} y\}.$$

We now claim that

$$(\forall x_2)(x_2 \in y \leftrightarrow (x_2 \leq x_3 \wedge \varphi(x_2,...,x_k)))$$

holds in M#. To see this, let $x_2 \in D$, $x_2 \in_{M\#} y$. Then $x_2 \in S$, and so $x_2 \leq x_3$, and $\varphi(x_2,...,x_k)$ holds in M#.

Conversely, suppose $x_2 \in D$, $x_2 \leq x_3$, and $\varphi(x_2,...,x_k)$ holds in M#. Then $x_2 \in S$, and so $x_2 \in_{M\#} y$.

For axiom 5, let $r \geq 1$, $\varphi(v_1,...,v_{2r})$ be a formula in L#. Let $1 \leq i_1,...,i_{2r}$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and the same min. Let $x_{r+1},...,x_{2r} \in D$, $x_{r+1},...,x_{2r} \leq \min(d_{i\_1},...,d_{i\_r})$.

Let $\varphi'(x_1,...,x_{2r+1})$ be given by Lemma 5.6.16. Then for all $x_1,...,x_{2r} \in D$,

$$\varphi(d_{i\_1},...,d_{i\_r},x_{r+1},...,x_{2r}) \text{ holds in } M\# \leftrightarrow$$

$\varphi'(c_{i\_1+1},...,c_{i\_r+1},x_{r+1},...,x_{2r},c_1)$ holds in M*.

$\varphi(d_{i\_r+1},...,d_{i\_2r},x_{r+1},...,x_{2r})$ holds in M# $\leftrightarrow$
$\varphi'(c_{i\_r+1+1},...,c_{i\_2r+1},x_{r+1},...,x_{2r},c_1)$ holds in M*.

By Lemma 5.5.8 viii), the right sides of the above two equivalences are equivalent. Hence the left sides of the above two equivalences are equivalent.

For axiom 6, NAT defines a nonempty initial segment under < by construction, and is I. I has no greatest element, and no limit point by Lemmas 5.6.8, 5.6.9. NAT lives below d_1 since $I \subseteq [0,c_2)$, according to Lemma 5.6.8, and $d_1 = c_2$. The first two elements of NAT are the 0,1 of M# by construction.

For axiom 7, by Lemma 5.6.8, I is internally well ordered in M*. By Lemma 5.6.16, NAT = I remains internally well ordered in M#.

For axiom 8, NAT with the $0,1,<,+,-,\bullet,\uparrow,\log$ of M# is the same as M(I). By Lemma 5.6.13, M(I) satisfies $TR(\Pi^0_1,L)$. Hence NAT with the $0,1,<,+,-,\bullet,\uparrow,\log$ of M# satisfies the sentences in $TR(\Pi^0_1,L)$.

The d's are unbounded in M# because the c's are unbounded in M*. QED

We now put Lemma 5.6.17 into our usual format to be used in the next section.

LEMMA 5.6.18. There exists a countable structure M# = $(D,<,\in,NAT,0,1,+,-,\bullet,\uparrow,\log,d_1,d_2,...)$ such that the following holds.
i) < is a linear ordering (irreflexive, transitive, connected);
ii) $x \in y \rightarrow x < y$;
iii) The d_n, $n \geq 1$, form a strictly increasing sequence of elements of D with no upper bound in D;
iv) Let φ be a formula of L# in which v_1 is not free. Then $(\exists v_1)(\forall v_2)(v_2 \in v_1 \leftrightarrow (v_2 \leq v_3 \wedge \varphi))$;
v) Let $r \geq 1$ and $\varphi(v_1,...,v_{2r})$ be a formula of L# not mentioning any constants d_n, $n \geq 1$. Let $1 \leq i_1,...,i_{2r}$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and min. Let $y_1,...,y_r \leq \min(d_{i\_1},...,d_{i\_r})$. Then $\varphi(d_{i\_1},...,d_{i\_r},y_1,...,y_r) \leftrightarrow \varphi(d_{i\_r+1},...,d_{i\_2r},y_1,...,y_r)$;

vi) NAT defines a nonempty initial segment under <, with no greatest element, and no limit point, where all points are $< d_1$, and whose first two elements are 0,1, respectively;
vii) ($\forall$x)(if x has an element obeying NAT then x has a < least element);
viii) Let $\varphi \in TR(\Pi^0_1, L)$. The relativization of φ to NAT holds.
ix) +,-,•,$\uparrow$,log have the default value 0 in case one or more arguments lie outside NAT.

Proof: This is immediate from Lemma 5.6.17. QED