5.8. ZFC + V = L, indiscernibles, and Π^0_1 correct arithmetic.

We fix M^ = (C,<,0,1,+,−,•,↑,log,ω,c_1,c_2,...,Y_1,Y_2,...) as given by Lemma 5.7.30. We work entirely within M^. E.g., we treat C as the universe of points, and regard the elements of the Y_k as the internal relations.

In particular, if we say that R is an internal relation, then we mean that R is an element of some Y_k. If we say that R is an M^ definable relation (first and second order parameters allowed), then we do not necessarily mean that R is an internal relation. However, by Lemma 5.7.30, vii), if R is an M^ definable relation which is **bounded**, then R is an internal relation; i.e., R is an element of some Y_k. In fact, Y_k is the set of all bounded M^ definable relations on C.

DEFINITION 5.8.1. Functions are always identified with their graphs. We refer to the elements of Y_1 as the internal sets.

An important obstacle is that there is no way of showing, in M^, that the family of all internal subsets of an internal set is in any sense internal. E.g., no way of showing that they are all cross sections of some fixed internal binary relation.

It would appear that this obstacle is fatal, as it indicates an inability to interpret the power set axiom, despite bounded comprehension, indiscernibility, and infinity.

However, in this section, we argue carefully that we can still construct the constructible universe. Because of the explicitness of this construction, we can use indiscernibility to overcome this obstacle within the constructible universe.

We first have to develop a pairing function. By an interval, we mean a set [x,y), where x,y ∈ C.

LEMMA 5.8.1. Let k ≥ 1 and F be a k-ary M^ definable function, defined without second order parameters. For all x, $\{F(y_1,...,y_k): y_1,...,y_k < x\}$ is bounded above. For all x, the restriction of F to $[0,x)^k$ is an internal function.

Proof: Let k,F be as given, and let $x \in C$. Let $n \geq 1$ be such that x and all parameters used in the definition of F are $< c_n$. Let $y_1,...,y_k < x$. Let $m > n$ be such that $F(y_1,...,y_k) < c_m$. Consider the true statement

$$F(y_1,...,y_k) < c_m.$$

This is a statement involving c_m and certain parameters $< c_n$. By Lemma 5.7.30 ix),

$$F(y_1,...,y_k) < c_{n+1}.$$

The second claim follows immediately by Lemma 5.7.30 vii). QED

DEFINITION 5.8.2. For all $x \in C$, we write x+1 for the immediate successor of x in <.

The above exists by Lemma 5.7.30 v),viii). This is a slight abuse of notation since x+1 already has a meaning, as + is a primitive of M^. However, note that by Lemma 5.7.30 ii),iii), if $x < \omega$ then x+1 is also the immediate successor of x in <.

LEMMA 5.8.2. Let $x,y \in C$, $x > 0$. There is a unique strictly increasing internal f with dom(f) = [0,x), rng(f) an interval, and f(0) = y.

Proof: We first prove a strong form of uniqueness. Suppose $x,x',y \in C$, $x,x' > 0$, and let f,g be strictly increasing internal functions, where dom(f) = [0,x), dom(g) = [0,x'), and rng(f),rng(g) are intervals, and f(0) = g(0) = y. Then f,g agree on their common domain. To see this, suppose this is false. By Lemma 5.7.30 viii), let b be < least such that $f(b) \neq g(b)$. Obviously f(b) is the strict sup of the f(c), c < b, and g(b) is the strict sup of the g(c), c < b. Hence f(b) = g(b), which is a contradiction. Hence f,g agree on their common domain.

For existence, fix $x,y \in C$. We prove that for all $0 < u \leq x$, there exists strictly increasing internal f such that dom(f) = [0,u), rng(f) is an interval, and f(0) = y.

Suppose this is false. By Lemma 5.7.30 viii), let $0 < u \leq x$ be < least such that this is false. For each $0 < v < u$, let f_v be the unique internal function which is strictly increasing with $dom(f_v) = [0,v)$, $rng(f_v)$ an interval, and $f(0) = y$.

First suppose u is a limit. By the comparability, the union, f, of the f_v, $v < u$, is a function M^ definable without second order parameters (but with second order quantifiers). Hence f is strictly increasing, with dom(f) = [0,u), f(0) = y. By Lemma 5.7.30 viii), rng(f) must be an interval. We have now contradicted the choice of u.

Now suppose u = v+1. If v = 0 then f_u obviously exists. Hence v > 0. Let f_v have range [y,z). Extend f_v to f by setting f(v) = z. Again we have contradicted the choice of u. QED

DEFINITION 5.8.3. We now define (x,y) <* (z,w) if and only if

i) max(x,y) < max(z,w); or
ii) max(x,y) = max(z,w) and (x,y) lexicographically precedes (z,w).

LEMMA 5.8.3. Every M^ definable binary relation R that holds of some (x,y), x,y ∈ C, holds of a <* least (x,y).

Proof: Let R be as given. The set of all max's of pairs at which R holds is obviously a nonempty M^ definable subset of C. By Lemma 5.7.30 viii), let u be its < least element. By Lemma 5.7.30 viii), let x be the < least first term of a pair at which R holds, whose maximum is u. By Lemma 5.7.30 viii), let y be the < least second term of a pair at which R holds, whose maximum is u, and whose first term is x. Then (x,y) is as required. QED

LEMMA 5.8.4. There is an M^ definable binary function $F:C^2 \rightarrow C$, defined without parameters, such that for all x,y ∈ C, F(x,y) is the strict sup of all F(z,w) with (z,w) <* (x,y). F is unique.

Proof: Define Q(u,G) if and only if

1) $G:\{x: x < u\}^2 \rightarrow C$ is internal, such that for all x,y < u, G(x,y) is the strict sup of all G(z,w), (z,w) <* (x,y).

We claim that for all v,w < u, if Q(v,G) and Q(w,H), then G,H agree on their common domain. This is proved in the obvious way using Lemma 5.8.3.

Define R(u) ↔ (∃G)(Q(u,G)). Suppose (∃u)(¬R(u)).

By Lemma 5.7.30 viii), let u be < least such that ¬R(u). Then for all v < u, there exists G with Q(v,G).

We thus see that for all v < u, there is a unique G_v with $Q(v,G_v)$, and these various G_v, v < u, are comparable.

Obviously R(0), and so u > 0.

Suppose u is a limit. By comparability, the union of the G_v, v < u, is an internal function G according to Lemma 5.8.1. It is obvious that Q(u,G). This contradicts the choice of u.

Now suppose u = v+1. We will extend G_v to G as follows. Since G_v is internal, by Lemma 5.7.30 viii), let u_1 be the strict sup of the values of G_v. By Lemma 5.8.2, let H be a strictly increasing internal function that maps [0,v) onto $[u_1,u_2)$, and J be a strictly increasing internal function that maps [0,v] onto $[u_2,u_3]$. Now extend G_v to G by defining G(w,v) = H(w) and G(v,w) = J(w), where w ≤ v. Clearly Q(u,G). This contradicts the choice of u.

We have thus established that for all u, R(u) holds.

We now define F as follows. Let x,y ∈ C. Let G be the unique internal function given by R(u), with u = max(x,y)+1. Set F(x,y) = G(x,y). It is clear that F is as required. F is unique by Lemma 5.8.3. QED

DEFINITION 5.8.4. We write P for the F constructed in the proof of Lemma 5.8.4.

LEMMA 5.8.5. For all x ∈ C, x ≤ P(0,x). Let x,y ∈ C. x > 0 → x,y < P(x,y). x,y ≤ P(x,y). $P:C^2 \rightarrow C$ is a bijection.

Proof: Suppose the first claim is false. By Lemma 5.7.30 viii), let x be < least such that P(0,x) < x. Then for all z < x, z ≤ P(0,z). Hence x ≤ P(0,x), which is a contradiction.

For the second claim, let x > 0. We have y ≤ P(0,y) < P(x,y), and x ≤ P(0,x) < P(x,0) ≤ P(x,y).

The third claim follows from the first two claims.

To see that P is one-one, let $P(x,y) = P(x',y')$. If $(x,y) <^* (x',y')$ then $P(x,y) < P(x',y')$. If $(x',y') <^* (x,y)$ then $P(x',y') < P(x,y)$. Hence $(x,y) = (x',y')$.

To see that P is onto, let x be the least element of C that is not a value of P. By the first claim, if $P(y,z) < x$ then $(y,z) <^* (0,x)$. It is easy to see that the strict sup of the (y,z) with $P(y,z) < x$ exists. Then the value of P at this strict sup must be x. Hence x is a value of P. This is a contradiction. QED

DEFINITION 5.8.5. We inductively define $P(x_1,\ldots,x_{k+1}) = P(P(x_1,x_2),x_3,\ldots,x_{k+1})$, for $k \geq 1$. Also define $P(x) = x$. This is our mechanism for coding sequences of points of standard finite length as points.

LEMMA 5.8.6. In each arity $k \geq 1$, P is a bijection. For all $k \geq 1$, $(\forall x_1,\ldots,x_k)(x_1,\ldots,x_k \leq P(x_1,\ldots,x_k))$. For all $k,n \geq 1$, $(\forall x_1,\ldots,x_k)(x_1,\ldots,x_k \leq c_n \rightarrow P(x_1,\ldots,x_k) < c_{n+1})$.

Proof: The first claim is proved by external induction on the arity, using that $P:C \rightarrow C$ and $P:C^2 \rightarrow C$ are bijections.

The second claim is proved by external induction on $k \geq 1$, using Lemma 5.8.5.

For the third claim, let $k,n \geq 1$ and $x_1,\ldots,x_k \leq c_n$. By Lemma 5.7.30 v), let $m > n$ be such that $P(x_1,\ldots,x_k) < c_m$. By Lemma 5.7.30 ix), $P(x_1,\ldots,x_k) < c_{n+1}$. QED

LEMMA 5.8.7. Let $k \geq 1$ and $R \subseteq C^k$. Then R is an internal relation if and only if $\{P(x_1,\ldots,x_k): R(x_1,\ldots,x_k)\}$ is an internal set.

Proof: Let k,R be as given. In the interest of caution, rewrite this set as

$$A = \{y: (\exists x_1,\ldots,x_k)(y = P(x_1,\ldots,x_k) \wedge R(x_1,\ldots,x_k))\}.$$

Suppose R is an internal relation; i.e., $R \in Y_k$. Then R is bounded. Hence by Lemma 5.8.1, A is bounded. By Lemma 5.7.30 vii), A is an internal set; i.e., $A \in Y_1$.

Now suppose A is an internal set. Then A is bounded. Hence by Lemma 5.8.6, R is bounded.

We claim that for all $x_1,...,x_k \in C$,

$$R(x_1,...,x_k) \leftrightarrow P(x_1,...,x_k) \in A.$$

To see this, suppose $R(x_1,...,x_k)$. Then $P(x_1,...,x_k) \in A$. Suppose $P(x_1,...,x_k) \in A$. Let $x_1',...,x_k'$ be such that $P(x_1,...,x_k) = P(x_1',...,x_k') \wedge R(x_1',...,x_k')$. Since P is one-one, we have $x_1 = x_1',...,x_k = x_k'$, and $R(x_1,...,x_k)$. Hence R is an internal relation, by Lemma 5.7.30 vii). QED

LEMMA 5.8.8. Any definable subset of C that contains 0 and is closed under +1, contains all $x < \omega$.

Proof: Let $B \subseteq C$ be definable, contain 0, and be closed under +1. Let $x < \omega$, $x \notin B$. By Lemma 5.7.30 viii), let x be least such that $x < \omega$, $x \notin B$. Then $x > 0$. By Lemma 5.7.30 iii), we have $x-1 < x$, and hence $x-1 \in B$. Therefore $x \in B$, which is a contradiction. QED

Lemma 5.8.8 supports proof by internal induction on $x < \omega$.

DEFINITION 5.8.6. An internal finite sequence is an internal function whose domain is some $[1,x]$, $x < \omega$.

We can use P to code internal finite sequences (from C) of indefinite length, as a single element of C.

LEMMA 5.8.9. Let $f:[1,x] \rightarrow C$, $x < \omega$, be internal. There exists a unique internal $g:[1,x] \rightarrow C$ such that for all $1 \leq u < x$,
i) $g(1) = f(1)$;
ii) $g(u+1) = P(g(u),f(u+1))$.
For this g, we have $g(x) \geq \max(f)$.

Proof: Let f,x be as given. We prove by internal induction on $z \leq x$, that there is an internal $g:[1,z] \rightarrow C$ such that for all $1 \leq u < z$, clauses i) and ii) hold. Internal induction below ω is supported by Lemma 5.8.8. The uniqueness of g can also be obtained using internal induction.

Clearly $\max(f)$ exists by induction. Also by induction, for all $1 \leq u \leq v \leq x$, $g(v) \geq f(u)$. Hence $g(x) \geq \max(f)$. QED

We use Lemma 5.8.9 to code finite sequences. Let $f:[1,x] \rightarrow C$, $x < \omega$.

DEFINITION 5.8.7. Define #(f) = P(x,g(x))+1, where g is given by Lemma 5.8.9. For empty f, define #(f) = 0.

LEMMA 5.8.10. For all internal finite sequences f,f', if #(f) = #(f') then f = f'.

Proof: Let f,f' be internal finite sequences. Let f:[1,x) → C, f':[1,y) → C. Suppose #(f) = #(f'). If x = 0 ∨ y = 0 then #(f) = #(f') = 0, and hence x = y = 0. So we assume that x,y > 0.

Let g,g' be given by Lemma 5.8.9, for f,f', respectively. Then #(f) = #(f') = P(x,g(x))+1 = P(y,g'(y))+1. Hence P(x,g(x)) = P(y,g'(y)), x = y, g(x) = g'(y). Hence g(x) = g'(x).

We now prove that f = f'. The case x = 1 is immediate, so we assume x > 1.

We first prove by reverse induction that for all 1 < x′ ≤ x, f(x′) = f'(x′) ∧ g(x′) = g'(x′). The basis case is x′ = x. By Lemma 5.8.9, we have g(x) = P(g(x-1),f(x)), g'(x) = P(g'(x-1),f'(x)). Hence f(x) = f'(x) ∧ g(x) = g'(x).

Suppose 2 < x' ≤ x, f(x') = f'(x'), g(x') = g'(x'). By Lemma 5.8.9, g(x') = P(g(x'-1),f(x')), g'(x') = P(g'(x'-1),f'(x')). Then g(x'-1) = g'(x'-1). By Lemma 5.8.9, g(x'-1) = P(g(x'-2),f(x'-1)), g'(x'-1) = P(g'(x'-2),f'(x'-1)). Hence f(x'-1) = f'(x'-1). This establishes the induction step.

So we have shown that for all 1 < x′ ≤ x, f(x′) = f'(x′) ∧ g(x′) = g'(x′). Hence f(2) = f'(2), g(2) = g'(2). By Lemma 5.8.9, g(1) = g'(1) = f(1) = f'(1). Hence f = f'. QED

LEMMA 5.8.11. (∀x)(∃y > x,ω)(∀z,w ≤ x)(P(z,w) < y).

Proof: Let x ∈ C. By Lemma 5.7.30 v), $\omega < c_1$, and we can let $x \leq c_n$. By Lemma 5.8.6, for all k ≥ 1 and $z_1,\ldots,z_k \leq c_n$, $P(z_1,\ldots,z_k) < c_{n+1}$ and $\omega < c_{n+1}$. QED

LEMMA 5.8.12. (∀x)(∃y > x,ω)(∀z,w < y)(P(z,w) < y).

Proof: Let x be given. Let u = max(x,ω). Informally, we want to construct u < P(u,u) < P(P(u),P(u)) < ... and take the sup. We can obviously prove by internal induction (Lemma 5.8.8) that for all n < ω, there exists unique internal $f_n:[1,n] \to C$ such that $f_n(1) = u$, and for all $1 \leq m < n$, $f_n(m+1) = P(f_n(m),f_n(m))$. By internal induction, these f_n are comparable, and so we can form their union F as an M^ definable function F with domain $\{n: 0 < n < \omega\}$.

By Lemma 5.8.1, F is an internal function. Also by internal induction, for all $0 < n < \omega$,

$$F(0) = u,\ F(n+1) = P(F(n),F(n)).$$

By Lemma 5.7.30 viii), let the strict sup of the values of F be y. We claim that

$$(\forall z,w < y)(P(z,w) < y).$$

Let $z,w < y$. Let $z,w \leq F(n)$. Then

$$P(z,w) \leq P(F(n),F(n)) = F(n+1) < y.$$

QED

According to Lemma 5.8.12, for $x \in C$, we let $P^*(x)$ be the least $y > x,\omega$ such that for all $z,w < y$, $P(z,w) < y$.

LEMMA 5.8.13. Let f be an internal finite sequence, $rng(f) \subseteq [0,x]$. Then $max(f) < \#(f) < P^*(x)$.

Proof: Let f be as given. If f is empty then $\#(f) = 0$ and we are done. We can assume that $f:[1,n] \to [0,x]$, where $1 \leq n < \omega$. Then $\omega < P^*(x)$. Let g be given by Lemma 5.8.9. By internal induction, for all $1 \leq i \leq n$, $g(i) < P^*(x)$. Hence $P(n,g(n)) < P(\omega,g(n)) < P^*(x)$. Therefore $\#(f) = P(n,g(n))+1 \leq P(\omega,g(n)) < P^*(x)$. By Lemma 5.8.9, $max(f) \leq g(n) < \#(f) < P^*(x)$. QED

We will need a notation for reverse finite sequence coding.

DEFINITION 5.8.8. Let $y \in C$ and $1 \leq i,n < \omega$. We define $y[i:n]$ to be the i-th term in the finite sequence of length n coded by y, if this exists; undefined otherwise. I.e., $y[i:n]$ is $f(i)$,

where $i \leq n$ and f is such that
$f:[1,n] \to C$, $\#(f) = y$,
provided f exists;
undefined otherwise.

By Lemma 5.8.10, the choice of f here, if it exists, is unique.

LEMMA 5.8.14. x[i:n] forms an M^ definable partial function from C × $[0,\omega)^2$ into C without parameters. Let f:[1,n] → C be internal, $1 \leq n < \omega$. The maximum value max(f) of f exists. There exists a unique x such that for all $i \leq n$, f(i) = x[i:n]. max(f) < x < P*(max(f)).

Proof: The first claim is obvious from the internal definition of x[i:n] above.

Now let f:[1,n] → C. For the second claim, an easy induction, using Lemma 5.7.30 i)-iii), shows that for all 1 ≤ i ≤ n, the maximum value of f on [0,i] exists.

By definition, #(f) = P(n,u)+1, for some u. Obviously u is unique, and we set x = #(f). Since rng(f) ⊆ [0,max(f)], we have max(f) < x = #(f) < P*(max(f) by Lemma 5.8.13. QED

M^, with its internal well foundedness (Lemma 5.7.30 viii)) and bounded comprehension (Lemma 5.7.30 vii)), is a relatively familiar context in which to work, compared with the earlier contexts in this chapter.

In order to construct the constructible hierarchy, we will use the usual language of set theory, L(∈,=).

DEFINITION 5.8.9. We take L(∈,=) to be based on ∈,=, variables v_n, $n \geq 1$, and ¬,∧,∀.

By the internal induction in Lemma 5.8.8, and Lemma 5.7.30 iii), we take internal arithmetic for granted, formulated on $[0,\omega)$.

In particular, we have access to the internal set GN of all Gödel numbers of formulas of L(∈,=).

DEFINITION 5.8.10. Let R be an internal binary relation. We let R# = P*(y), where y is least such that (∀x ∈ fld(R))(x < y).

The idea is that R# is large enough to accommodate all of the internal finite sequence codes that we need, in the sense of Lemma 5.8.14.

We wish to formally define the notion SAT(R,n,x,m).

DEFINITION 5.8.11. The intended meaning of SAT(R,n,x,m) is that

i) R is a binary relation;
ii) $n \in GN$, $x < R\#$;
iii) the subscript of every free variable in the formula φ of $L(\in,=)$ with Gödel number n is $\leq m < \omega$;
iv) (fld(R),R) satisfies φ at the partial assignment x[1:m],x[2:m],...,x[m:m].

Note that we allow R to be empty.

In order for clause iv) to hold, we require that x[1:m],x[2:m],...,x[m:m] $\in$ fld(R).

Note that if m = 0 then the partial assignment in clause iv) is empty.

In order to make this definition over M^, we first need the following.

LEMMA 5.8.15. Let R be an internal binary relation. There exists a unique internal ternary relation $SAT_R \subseteq GN \times [0,R\#) \times [0,\omega)$ satisfying the usual Tarski satisfaction conditions.

Proof: Let R be as given. Note that in M^, the code of every finite length sequence form fld(R) is < R#, by Lemma 5.8.14. The uniqueness of $SAT_R(n,x,m)$ is proved by internal induction on n. For existence, prove by internal induction on $r < \omega$ that there is a ternary relation $T_r \subseteq GN|r \times [0,R\#) \times [0,\omega)$, that satisfies the usual Tarski satisfaction conditions for all $n \in GN|r$. Here GN|r is the set of all $n \in GN$ which is at most r. Also prove by internal induction on $r < \omega$ that each T_r is unique, and the T_r's are compatible, in the sense that they agree on their common domain. Furthermore, each $T_r \subseteq [0,R\#]^3$. By Lemma 5.7.30, we can take SAT_R to be the union of the T_r's. Finally, an internal induction shows that SAT_R is unique. QED

DEFINITION 5.8.12. We now define SAT(R,n,x,m) if and only if R is a binary relation, and $SAT_R(n,x,m)$ holds, where $SAT_R(n,x,m)$ is given by Lemma 5.8.15.

DEFINITION 5.8.13. Let R be an internal binary relation. We say that n,x,m is a code over R if and only if

i) $n \in GN$;
ii) $1 \leq m < \omega$;
iii) $x < R\#$ is greater than all elements of fld(R).

We remark that condition iii) is convenient because x does not interfere with the elements of fld(R).

DEFINITION 5.8.14. If n,x,m is a code over R then we write H(R,n,x,m) for

$$\{y: (\exists z)(z[1:m] = y \wedge z[2:m] = x[2:m] \wedge \ldots \wedge z[m:m] = x[m:m] \wedge SAT(R,n,z,m))\}.$$

Note that in the above definition, we use x[2:m],...,x[m:m] but not x[1:m]. This means that we can easily modify x without changing H(R,n,x,m). We will exploit this freedom below.

We think of H(R,n,x,m) as the internal subset of fld(R) that is coded by the code n,x,m. Informally, the H(R,n,x,m), where n,x,m is a code over R, code exactly the "subsets of fld(R) that are first order definable over R". The case $R = \emptyset$ is handled appropriately with this notation.

DEFINITION 5.8.15. We say that n,x,m is a minimal code over R if and only if n,x,m is a code over R such that

i) for all codes n′,x′,m′ over R, if H(R,n′,x′,m′) = H(R,n,x,m) then $P(n,x,m) \leq P(n',x',m')$;
ii) for all $y \in$ fld(R), $H(R,n,x,m) \neq \{z: R(z,y)\}$.

Thus the minimal codes over R code exactly the R definable subsets of fld(R) that are not already of the form $\{z: R(z,y)\}$, $y \in$ fld(R). Also, by minimality, no two distinct minimal codes over R code the same subset of fld(R).

Minimal codes are preferred codes used in order to ensure the propagation of extensionality as we construct the constructible hierarchy.

LEMMA 5.8.16. Let $\varphi(v_1,\ldots,v_m)$, $m \geq 1$, be a formula of $L(\in,=)$ with Gödel number n. Let R be an internal binary relation. Then SAT(R,n,x,m) holds if and only if $\varphi(x[1:m],\ldots,x[m:m])$ holds in (fld(R),R). $H(R,n,x,m) = \{y: \varphi(y,x[2:m],\ldots,x[m:m])$ holds in (fld(R),R)$\}$.

Proof: Left to the reader. Note that φ,m,n are standard. QED

LEMMA 5.8.17. Let $\varphi(v_1,\ldots,v_m)$, $m \geq 1$, be a formula of $L(\in,=)$. Let R be an internal binary relation, and $z_1,\ldots,z_{m-1} \in fld(R)$. Then $\{y: \varphi(y,z_1,\ldots,z_{m-1})$ holds in $(fld(R),R)\}$ is either of the form $\{y: R(y,x)\}$, $x \in fld(R)$, or of the form $H(R,n',x',m')$, for some unique minimal code n',x',m' over R, but not both.

Proof: Use Lemma 5.8.16. Note that φ,m,n are standard. Assume that

$$\{y: \varphi(y,z_1,\ldots,z_{m-1}) \text{ holds in } (fld(R),R)\}$$
$$\text{is not of the form } \{y: R(y,x)\},\ x \in fld(R).$$

Let $f:[1,m] \to C$, where $f(2) = z_1,\ldots,f(m) = z_{m-1}$, and where f(1) is the least point greater than all elements of fld(R). Let $x = \#f$. Then $x < R\# = P^*(f(1))$, is greater than all elements of fld(R), and

$$H(R,n,x,m) = \{y: \varphi(y,z_1,\ldots,z_{m-1}) \text{ holds in } (fld(R),R)\}.$$

So we can minimize over the relevant n,x,m in order to obtain the required minimal code n',x,m' over R. By the definition of minimal codes over R, the or is exclusive. QED

We are now ready to construct the binary relation FODO(R), for internal R, obtained by "adjoining" all sets first order definable over (fld(R),R) to R.

DEFINITION 5.8.16. We say that a binary relation R is adequate if and only if

$$R(0,1) \wedge (\forall x)(\neg R(x,0)).$$

In particular, for adequate R, we have $0,1 \in fld(R)$.

For internal adequate binary relations R, we construct FODO(R) as follows.

DEFINITION 5.8.17. We define FODO(R)(u,v) if and only if either R(u,v), or

i) there exists a minimal code n,x,m over R such that $v = P(n,x,m)$;

ii) $u \in H(R,n,x,m)$.

The reason that we need the adequacy of R is that $\emptyset = \{x: R(x,0)\}$, and so there is no minimal code n,x,m over R with $H(R,n,x,m) = \emptyset$. It will be convenient to have the sets with minimal codes over R be nonempty.

DEFINITION 5.8.18. Let R be an internal binary relation. We say that R is extensional if and only if for all $x,y \in fld(R)$, $(\forall z)(R(z,x) \leftrightarrow R(z,y)) \rightarrow x = y$.

DEFINITION 5.8.19. We say that a binary relation R is sharply extended by a binary relation S if and only if

i) $(\forall x \in fld(S)\backslash fld(R))(\forall y \in fld(R))(y < x)$;
ii) $(\forall x,y \in fld(R))(R(x,y) \leftrightarrow S(x,y))$.
iii) $S(x,y) \wedge y \in fld(R) \rightarrow x \in fld(R)$.
iv) fld(R) is a proper subset of fld(S).

LEMMA 5.8.18. Let R be an internal adequate binary relation. Then FODO(R) is an internal adequate binary relation. In addition, R extensional $\rightarrow$ FODO(R) extensional. FODO(R) sharply extends R. $(\forall x,y)(R(x,y) \rightarrow x < y) \rightarrow (\forall x,y)(FODO(R)(x,y) \rightarrow x < y)$.

Proof: Let R be as given. Note that FODO(R) is bounded by R#. By Lemma 5.6.30 vii), FODO(R) is internal. We claim that

$$1)\ v \in fld(R) \rightarrow FODO(R)(u,v) \leftrightarrow R(u,v).$$

$$2)\ FODO(R)(u,v) \rightarrow u \in fld(R).$$

For 1), let $v \in fld(R)$. If $R(u,v)$ then $FODO(R)(u,v)$. Suppose $FODO(R)(u,v)$. Assume $\neg R(u,v)$. Let $v = P(n,x,m)$, where n,x,m is a minimal code over R. Then x is greater than all elements of fld(R). Hence $x > v$, which is impossible.

For 2), let $FODO(R)(u,v)$. If $R(u,v)$ then obviously $u \in fld(R)$. So we can let $v = P(n,x,m)$, n,x,m a minimal code over R. Then $u \in H(R,n,x,m) \subseteq fld(R)$.

FODO(R) is adequate since $R \subseteq FODO(R)$ and by 1), $FODO(R)(u,0) \rightarrow R(u,0)$, which is impossible.

Assume R is extensional. We claim that FODO(R) is extensional. Suppose

$$3)\ (\forall x)(\mathrm{FODO}(R)(x,y) \leftrightarrow \mathrm{FODO}(R)(x,z)).$$

case 1. $y,z \in \mathrm{fld}(R)$. Since R is extensional, $y = z$.

case 2. $y,z \notin \mathrm{fld}(R)$. Let $y = P(n,x,m)$, $z = P(n',x',m')$, where n,x,m and n',x',m' are minimal codes over R. By 2),3), $H(R,n,x,m) = H(R,n',x',m')$. Hence $P(n,x,m) \leq P(n',x',m') \leq P(n,x,m)$. So $P(n,x,m) = P(n',x',m') = y = z$.

case 3. $y \in \mathrm{fld}(R)$, $z \notin \mathrm{fld}(R)$. Let $z = P(n,x,m)$, n,x,m a minimal code over R, $H(R,n,x,m) \neq \{z: R(z,y)\}$, $H(R,n,x,m)$, and $\{z: R(z,y)\} \subseteq \mathrm{dom}(R)$. This contradicts 3).

case 4. $y \notin \mathrm{fld}(R)$, $z \in \mathrm{fld}(R)$. This leads to a contradiction as in case 3.

We have thus derived $y = z$ from 3), and FODO(R) is extensional.

We claim that FODO(R) sharply extends R. For i) of the definition of sharply extended, let $x \in \mathrm{fld}(\mathrm{FODO}(R))\backslash\mathrm{fld}(R)$, $y \in \mathrm{fld}(R)$. Then $x = P(n,u,m)$, where n,u,m is a minimal code over R. Hence u is greater than all elements of fld(R), and so $x > y$.

For ii), use 1).

For iii), use 2).

For iv), note that $\{x \in \mathrm{fld}(R): \neg R(x,x)\}$ cannot be of the form $\{y: R(y,x)\}$, $x \in \mathrm{fld}(R)$. Let n,x,m be a minimal code over R such that $H(R,n,x,m) = \{x \in \mathrm{fld}(R): \neg R(x,x)\}$. THen $P(n,x,m) \in \mathrm{fld}(\mathrm{FODO}(R))\backslash\mathrm{fld}(R)$.

Hence by Lemma 5.8.17, $\mathrm{fld}(R) = H(R,n,x,m)$, for some minimal code n,x,m over R. Hence $\mathrm{fld}(R) = \{y: \mathrm{FODO}(R)(y,x)\}$. Therefore $\mathrm{fld}(R) \neq \mathrm{fld}(\mathrm{FODO}(R))$.

For the last claim, assume $(\forall x,y)(R(x,y) \rightarrow x < y)$. Let $\mathrm{FODO}(R)(x,y)$. By construction, either $R(x,y)$ or

$$x \in \mathrm{fld}(R) \wedge y \text{ is some } P(n,z,m),$$
where z is greater than all elements of fld(R).

In either case, $x < y$. QED

LEMMA 5.8.19. Let R be an internal adequate binary relation. Every set definable in (fld(R),R) is of the form {x: FODO(R)(x,y)}, where $y \in$ fld(FODO(R)).

Proof: By the construction of FODO(R), and Lemmas 5.8.17, 5.8.18. QED

Here we have interpreted Lemma 5.8.19 as a scheme of assertions about M^, where we take "definable" in the external sense. However, we also want to interpret Lemma 5.8.19 in a stronger, internal sense - using SAT_R from Lemma 5.8.15. This stronger form of Lemma 5.8.19 can also be proved with the help of internal inductions.

We now wish to transfinitely iterate the FODO operation. The base of the transfinite iteration will be the adequate relation

$$R_0(x,y) \leftrightarrow x = 0 \wedge y = 1.$$

In order to accomplish this, we must be a bit careful. Firstly, we must note that, conceptually, we are manipulating internal relations, and these internal relations are not points; they are elements of Y_2. Furthermore, these internal relations are not even coded as points. In contrast, recall that internal finite sequences f of points are coded as points using f#.

Secondly, note that the operation that sends appropriate R to FODO(R) is even further removed from being an object. It is merely a description of a relationship between objects (not even between points), given in a first order way, without parameters, over M^.

Our strategy is to properly define what we mean by a transfinite iteration of the operation up through a point, *as an object*. The objects for this purpose are the elements of the Y_k, $k \geq 1$. These are components of M^.

DEFINITION 5.8.20. Let T be a k+1-ary relation, $k \geq 1$. For $x \in C$, we write T_x for the cross section $\{(y_1,\ldots,y_k): T(x,y_1,\ldots,y_k)\}$.

Note that T_x is a k-ary relation.

LEMMA 5.8.20. Let $x \in C$. There is a unique internal ternary relation T such that
i) $T_0 = R_0$;
ii) For all $y < x$, $T_{y+1} = FODO(T_y)$;
iii) For all limits $y \leq x$, $T_y = \cup_{z<y} T_z$;
iv) For all $y \leq x$, T_y is adequate;
v) For all $y > x$, $T_y = \emptyset$.

Proof: Define $\Gamma(T,x)$ if and only if $x \in C \wedge T$ is an internal ternary relation obeying i)-v).

We first claim that for all x,T,T',

$$\Gamma(T,x) \wedge \Gamma(T',x) \rightarrow T = T'.$$

Suppose this is false. Choose x to be least such that

$$(\exists T,T')(\Gamma(T,x) \wedge \Gamma(T',x) \wedge T \neq T'.$$

Clearly $x \neq 0$, since $T_0 = T'_0 = R_0$. Hence $x > 0$.

Let $x = z+1$. Let

$$\Gamma(T,z+1),\ \Gamma(T',z+1), T_{z+1} \neq T'_{z+1}.$$
$$FODO(T_z) \neq FODO(T'_z).$$
$$T_z \neq T'_z.$$

This contradicts the choice of x.

Finally, let x be a limit. We claim that

$$(\forall z < x)(T_z = T'_z).$$

To see this, let $z < x$. Let T^* be the restriction of T to triples whose first argument is $\leq z$, and $T^{*\prime}$ be the restriction of T^* to triples whose first argument is $\leq z$. Then $\Gamma(T^*,z)$, $\Gamma(T^{*\prime},z)$. Hence $T^* = T^{*\prime}$. This is a contradiction.

The first claim has been established. In fact, it is now clear that the T's such that $(\exists x)(\Gamma(T,x))$ are comparable in that any two agree on their common domain.

To prove existence, let $u > 0$, and suppose

$$(\forall x < u)(\exists T)(\Gamma(T,x)).$$

We now show

$$(\exists T)(\Gamma(T,u)).$$

The case u = 0 is obvious, by defining

$$T(a,b,c) \leftrightarrow a = 0 \wedge R_0(b,c).$$

Assume u is a successor, u = v+1. Let $\Gamma(T,v)$. Define

$$T'(a,b,c) \leftrightarrow$$
$$T(a,b,c) \vee (a = v+1 \wedge \text{FODO}(T_v)(b,c)).$$

To see that T′ is internal, it suffices to show that T′ is bounded. This follows from the boundedness of T and $\text{FODO}(T_v)$.

Note that by Lemma 5.8.18, $\text{FODO}(T_v) = T'_{v+1}$ is adequate. Also,

$$x \leq v \rightarrow T'_x = T_x.$$
$$T'_{v+1} = \text{FODO}(T_v) = \text{FODO}(T'_v).$$
$$\Gamma(T',v+1),\ \Gamma(T',u).$$

Assume u is a limit. Define

$$T^*(a,b,c) \leftrightarrow$$
$$a < u \wedge (\exists T)(\Gamma(T,a) \wedge T(a,b,c)).$$

To see that T* is internal, it suffices to show that T* is bounded. We have $(\forall a < u)(\exists! T)(\Gamma(T,a))$, by the first claim (uniqueness).

Let $a < u < c_n$, $n \geq 1$. By Lemma 5.7.30 ix), we have

$$(\exists w)(\exists T)(\Gamma(T,a) \wedge T \text{ lies entirely below } w).$$
$$(\exists w < c_{n+1})(\exists T)(\Gamma(T,a) \wedge T \text{ lies entirely below } w).$$
$$(\exists T)(\Gamma(T,a) \wedge T \text{ lies entirely below } c_{n+1}).$$
$$T^* \text{ lies entirely below } c_{n+1}.$$
$$T^* \text{ is internal.}$$

Let $a < u$. Let

$$\Gamma(T,0),\ \Gamma(T,a),\ \Gamma(T',a+1).$$

From the definition of T* and the uniqueness/comparability (first claim),

$$T^*_0 = T_0,\ T^*_a = T_a = T'_a,\ T^*_{a+1} = T'_{a+1}.$$
$$T_0 = R_0,\ T^*_{a+1} = \mathrm{FODO}(T'_a) = \mathrm{FODO}(T^*_a).$$
All of these relations are adequate.

Now let z < u be a limit. Let $\Gamma(z,T'')$. Then

$$T''_z = T^*_z.$$
$$T''_z = \cup_{a<z} T''_a = \cup_{a<z} T^*_a.$$

Hence T* obeys i)-v) for $\Gamma(T^*,u)$, except clause iii) holds only for y < u. To fix this, define

$$T^{**}(a,b,c) \leftrightarrow$$
$$T^*(a,b,c) \lor (a = u \land (\exists a < u)(T^*(a,b,c))).$$

It is easy to see that T^{**}_u is adequate. Then $\Gamma(T^{**},u)$. QED

DEFINITION 5.8.21. For each x ∈ C, we let $L(x) = T_x$, where T is the ternary relation given by Lemma 5.8.20. I.e., where $\Gamma(T,x)$ as defined in the proof of Lemma 5.8.20. Thus each $L(x) \in Y_2$.

DEFINITION 5.8.22. For each x ∈ C, we define L[x] = fld(L(x)). Note that L[0] = {0,1}, and that L[x] ⊆ C.

DEFINITION 5.8.23. We define L[∞] as the union of the L[x].

We caution the reader that L[∞] ⊆ C is not internal, because it is not bounded. It is, however, M^ definable without any parameters.

DEFINITION 5.8.24. We define L(∞) be the union of the L(x).

Thus L(∞)(x,y) if and only if there exists z ∈ C such that L(z)(x,y). Obviously $L(\infty) \subseteq C^2$.

The various L[x] correspond to the initial segments of the constructible hierarchy. The various L(x) correspond to the epsilon relations on the initial segments of the constructible hierarchy. L[∞] corresponds to the class of constructible sets. L(∞) corresponds to the epsilon relation on the class of constructible sets.

Clearly $L(\infty)$ is the version of the epsilon relation on the constructible sets in M^, and is a binary relation. Its field is $L[\infty]$.

We caution the reader that L[x] may not be an initial segment of points, and may not be a subset of [0,x). It may have elements that are greater than x.

LEMMA 5.8.21. $L(0) = R_0$. For all $x \in C$, $L(x+1) = FODO(L(x))$. For all limits $x \in C$, L(x) is the union of the L(y), y < x. For all x < y, L(x) is sharply extended by L(y). Each L(x) is extensional. Each L(x) has $L(x)(y,z) \rightarrow y < z$.

Proof: L(0) = T, where $\Gamma(T,0)$. Hence $L(0) = R_0$. $L(x+1) = T_{x+1}$, where $\Gamma(T,x+1)$. Hence $L(x+1) = FODO(T_x)$. Let T′ be the restriction of T to triples whose first argument is $\leq$ x. Then $\Gamma(T',x)$, $T'_x = T_x$, $L(x+1) = FODO(T'_x) = FODO(L(x))$.

Let x be a limit. $L(x) = T_x$, where $\Gamma(T,x)$. Now $T_x = \cup_{y<x}T_y$. By using restrictions as in the previous paragraph, we see that for all y < x, $T_y = L(y)$. Hence $L(x) = T_x = \cup_{y<x}T_y$.

For the fourth claim, fix x. We prove by transfinite induction on y that

$$x < y \rightarrow L(x) \text{ is sharply extended by } L(y).$$

This is obvious for y = x.

Suppose y > x, and L(x) is sharply extended by L(y). By Lemma 5.8.18, L(y) is sharply extended by L(y+1). Since L(x) is sharply extended by L(y), clearly L(x) is sharply extended by L(y+1).

Suppose y > x, where y is a limit, and L(x) is sharply extended by every L(z), $x \leq z < y$. We claim that L(x) is sharply extended by L(y). To see this, first let $u \in fld(L(y))\backslash fld(L(x))$, $v \in fld(L(x))$. Let $u \in fld(L(z))\backslash fld(L(x))$, x < z < y. Since L(z) is sharply extended by L(x), we have u < v.

Next let $u,v \in fld(L(x))$. If L(x)(u,v) then obviously L(y)(u,v). If L(y)(u,v) then let x < z < y, L(z)(u,v). Since L(x) is sharply extended by L(z), we have L(x)(u,v).

Now let L(y)(u,v), v ∈ fld(L(x)). Let u ∈ L(z), where x < z < y. Since L(z) is sharply extended by L(y), L(z)(u,v). Since L(x) is sharply extended by L(z), we have u ∈ L(x).

Finally, fld(L(x)) is a proper subset of fld(L(y)) since fld(L(x)) is a proper subset of fld(L(x+1)) ⊆ fld(L(y)), by Lemma 5.8.18.

For the fifth claim, we argue by transfinite induction on x. L(0) = R_0 is extensional. Suppose L(x) is extensional. By Lemma 5.8.18, L(x+1) = FODO(L(x)) is extensional. Suppose x is a limit, where for all y < x, L(y) is extensional. Let a,b ∈ fld(L(x)), (∀z)(L(x)(z,a) ↔ L(x)(z,b)). Let a,b ∈ fld(L(y)), y < x. Since L(x) is a sharp extension of L(y), we have (∀z)(L(y)(z,a) ↔ L(y)(z,b)). Since L(y) is extensional, a = b.

For the sixth claim, we argue by transfinite induction on x. Obviously L(0)(y,z) → y < z since L(0) = R_0. Suppose

(∀y,z)(L(x)(y,z) → y < z).

By Lemma 5.8.18,

(∀y,z)(L(x+1)(y,z) → y < z).

Let x be a limit, where

(∀y < x)(∀u,v)(L(y)(u,v) → u < v).

Let L(x)(u,v). Let L(y)(u,v), y < x. Then u < v. QED

DEFINITION 5.8.25. Let x ∈ L[∞]. We write lrk(x) for the least y such that x ∈ L[y+1]. This is the L rank of x. Note that lrk is a function from C into C that is M^ definable without parameters.

LEMMA 5.8.22. Let x,y ∈ C. L(∞)(x,y) → (lrk(x) < lrk(y) ∧ x < y). L(∞)(x,y) ↔ L(lrk(y)+1)(x,y). L[∞] ∩ [0,x) ⊆ L[x].

Proof: Let L(∞)(x,y). Let L(z)(x,y). By Lemma 5.8.21, x < y. Also, let y ∈ L[u+1]\L[u]. Then lrk(y) = u, u+1 ≤ z. By Lemma 5.8.21, z = u+1 ∨ L(u+1) is sharply extended by L(z). Therefore x ∈ L(u+1), L(u+1)(x,y), x ∈ L(u). Hence lrk(x) < lrk(y) = u. This also establishes the second claim.

We prove the final claim by transfinite induction on x. We have $L[\infty] \cap [0,0) \subseteq L[0]$, vacuously.

Suppose $L[\infty] \cap [0,x) \subseteq L[x]$. We want $L[\infty] \cap [0,x] \subseteq L[x+1]$. It suffices to prove $x \in L[\infty] \rightarrow x \in L[x+1]$. Assume $x \in L[\infty]\backslash L[x+1]$. Let $x \in L[y]$. Then $y > x+1$. Since L[y] sharply extends L[x+1], x is greater than all elements of L[x+1]. Since L[x+1] sharply extends L[x], there is an element of L[x+1] that is greater than all elements of L[x], and $L[x] \supseteq [0,x)$. Hence there is an element of L[x+1] that is $\geq x$. This is a contradiction.

Suppose x is a limit, where for all $y < x$,

$$L[\infty] \cap [0,y) \subseteq L[y].$$

We claim that

$$L[\infty] \cap [0,x) \subseteq L[x].$$

To see this, let $z \in L[\infty]$, $z < x$. Let $z < y < x$. Then $z \in L[y]$, $z \in L[x]$. QED

DEFINITION 2.8.26. A Δ_0 formula of $L(\in,=)$ is a formula of $L(\in,=)$ in which all quantifiers are $\in$ bounded; i.e.,

$$(\exists x \in y)$$
$$(\forall x \in y)$$

where x,y are distinct variables.

LEMMA 5.8.23. Let $\varphi(x_1,...,x_k)$ be a Δ_0 formula of $L(\in,=)$. Let $y_1,...,y_k,z,w$ be such that $y_1,...,y_k \in L[z],L[w]$. Then $\varphi(y_1,...,y_k)$ holds in $(L[z],L(z))$ if and only if $\varphi(y_1,...,y_k)$ holds in $(L[w],L(w))$ if and only if $\varphi(y_1,...,y_k)$ holds in $(L[\infty],L(\infty))$.

Proof: Here k,φ are standard. The first claim is by external induction on the number of occurrences of variables in φ. Use Lemma 5.8.21 (sharp extensions). QED

LEMMA 5.8.24. Extensionality, pairing, and union hold in $(L[\infty],L(\infty))$.

Proof: For extensionality, let $x,y \in L[u]$, where $(\forall z)(z \in x \leftrightarrow z \in y)$ holds in $(L[\infty],L(\infty))$. By Lemma 5.8.23, $(\forall z)(z \in x$

$\leftrightarrow$ z $\in$ y) holds in (L[u],L(u)). By Lemma 5.8.21, x = y. Since u is arbitrary, extensionality holds in (L[∞],L(∞)).

For pairing, let x,y $\in$ L[u]. By Lemma 5.8.19, let z $\in$ L[u+1] be such that $(\forall w)(w \in z \leftrightarrow (w = x \vee w = y))$ holds in L[u+1]. By Lemma 5.8.21 (sharp extensions), $(\forall w)(w \in z \leftrightarrow (w = x \vee w = y))$ holds in (L[∞],L(∞)). Since u is arbitrarily, pairing holds in (L[∞],L(∞)).

For union, let x $\in$ L[u]. By Lemma 5.8.19, let y in L[u+1] be such that
$(\forall z)(z \in y \leftrightarrow (\exists w)(z \in w \wedge w \in x))$ holds in (L[u+1],L(u+1)). By Lemmas 5.8.21 (sharp extensions) and 5.8.23, $(\forall z)(z \in y \leftrightarrow (\exists w)(z \in w \wedge w \in x))$ holds in (L[∞],L(∞)). Since u is arbitrary, union holds in (L[∞],L(∞)). QED

LEMMA 5.8.25. Infinity holds in (L[ω+1],L(ω+1)). Infinity holds in (L[∞],L(∞)).

Proof: Infinity has the form

$$(\exists x)(\varnothing \in x \wedge (\forall y \in x)(y \cup \{y\} \in x))$$

which makes perfectly good sense in the presence of extensionality, union, and pairing. It is clear that 0 serves as the $\varnothing$ in (L[∞],L(∞)).

We say that a set is epsilon connected if and only if any two elements are either equal, or one is an element of the other.

Prove by internal induction on n < ω that "the epsilon connected transitive sets are linearly ordered by epsilon, and there is a largest epsilon connected transitive set" holds in (L[n],L(n)). For each n < ω, let h(n) be the witness to this statement in (L[n+1],L(n+1)). Prove by internal induction on n < ω that h(0) = $\varnothing$, and "h(n+1) = h(n) $\cup$ {h(n)}" holds in (L[n+2],L(n+2)). Prove that for all u $\in$ L(ω), $(\exists n < \omega)(u = h(n))$ if and only if "u is epsilon connected and transitive" holds in (L[ω],L(ω)). By Lemma 5.8.19, let x $\in$ L(ω+1), where $(\forall y)(L(\omega+1)(y,x) \leftrightarrow (\exists n < \omega)(y = h(n))$. Then in (L[$\omega$+1],L($\omega$+1)), x is a witness for Infinity.

To see that Infinity holds in (L[∞],L(∞)), apply Lemma 5.8.21, with parameters x,0. QED

LEMMA 5.8.26. Every L(x) is internally well founded. L(∞) is internally well founded. Foundation holds in every (L[x],L(x)). Foundation holds in (L[∞],L(∞)).

Proof: The first claim follows from the internal well foundedness of < by Lemma 5.8.21. The internal well foundedness of < is by Lemma 5.7.30 viii). The remaining claims follow easily from the first claim, using Lemma 5.8.23. QED

LEMMA 5.8.27. Let $n \geq 1$ and $\varphi_1,\ldots,\varphi_n$ be formulas of L(∈,=) that begin with, respectively, existential quantifiers $(\exists y_1),\ldots,(\exists y_n)$. For all z there exists w > z such that the following holds. Let $1 \leq i \leq n$. Let the free variables of φ_i be assigned elements of L[z]. If φ_i holds in (L[∞],L(∞)) then $(\exists y_i \in L[w])(\varphi_i(y_i))$ holds in (L[∞],L(∞)).

Proof: By Lemma 5.8.1, we can choose internal witness functions $f_1,\ldots,f_k$, whose domains are Cartesian powers of L[z]. By applying the lrk function to the values of the f's, we see that the set A of values of lrk(z), z a value of the f's, must be internal - again using Lemma 5.8.1. Take w to be the strict sup of A. QED

LEMMA 5.8.28. Let $\varphi(v_1,\ldots,v_k)$ be a formula of L(∈,=). For all z there exists w > z such that the following holds. Let $y_1,\ldots,y_k \in L[w]$. Then $\varphi(y_1,\ldots,y_k)$ holds in (L[∞],L(∞)) if and only if $\varphi(y_1,\ldots,y_k)$ holds in (L[w],L(w)).

Proof: Without loss of generality, we can assume that $\varphi(v_1,\ldots,v_k)$ is in prenex normal form. Let $\varphi_1,\ldots,\varphi_n$ be a listing of all direct subformulas of φ, and duals of subformulas of φ, which begin with an existential quantifier.

Informally, we define, internally, an infinite sequence $z < w_1 < w_2 < \ldots$ as follows. w_1 is the least w > z given by Lemma 5.8.27 for $\varphi_1,\ldots,\varphi_n$. Suppose w_j has been defined, $j \geq 1$. w_{j+1} is the least $w > w_j$ given by Lemma 5.8.27 with z set to w_j.

We convert this to a construction within M^ as follows. First prove that for all $n < \omega$, there is a unique finite sequence $f:[1,n] \to C$, where $f(1) = w_1$ and each f(i+1) is obtained from f(i) according to the previous paragraph. This yields a function $g:[1,\omega) \to C$ by taking the union of these f's. Now apply Lemma 5.8.1 to show that g is internal. In particular, g is bounded, and so we let w be the strict sup of the values of g.

An external induction argument shows that for all $y_1,\ldots,y_k \in L[w]$ and $1 \leq i \leq n$,

$$\varphi_i(y_1,\ldots,y_k) \text{ holds in } (L[\infty],L(\infty)) \leftrightarrow$$
$$\varphi_i(y_1,\ldots,y_k) \text{ holds in } (L[w],L(w)).$$

The induction is on the number of quantifiers present in φ_i. Since φ is among the $\varphi_1,\ldots,\varphi_n$, we are done. QED

DEFINITION 2.8.27. Collection is the scheme

$$(\forall x \in y)(\exists z)(\varphi) \rightarrow (\exists w)(\forall x \in y)(\exists z \in w)(\varphi)$$

where φ is a formula of $L(\in,=)$, x,y,z,w are distinct variables, and w is not free in φ.

LEMMA 5.8.29. Every instance of Separation holds in $(L[\infty],L(\infty))$. Every instance of Collection holds in $(L[\infty],L(\infty))$.

Proof: Consider $(\exists x)(\forall y)(y \in x \leftrightarrow (y \in z \wedge \varphi))$, where x,y,z are distinct variables and x is not free in φ. Let $z \in L[\infty]$. Let u be such that z and all parameters in φ lie in L[u].

By Lemma 5.8.28, let v > u be such that for all $y \in L[v]$,

$$\varphi(y) \text{ holds in } (L[\infty],L(\infty)) \leftrightarrow$$
$$\varphi(y) \text{ holds in } (L[v],L(v)).$$

Let $b \in L[v+1]$, where

$$(\forall y)(L(\infty)(y,b) \leftrightarrow$$
$$((y \in z \wedge \varphi(y)) \text{ holds in } (L[v],L(v))).$$

Then

$$(\forall y)(y \in b \leftrightarrow (y \in z \wedge \varphi))$$

holds in $(L[\infty],L(\infty))$.

Now consider

$$(\forall x \in y)(\exists z)(\varphi) \rightarrow (\exists w)(\forall x \in y)(\exists z \in w)(\varphi),$$

where x,y,z,w are distinct variables and w is not free in φ. Let y ∈ L[∞]. Let u be such that y and all parameters in φ lie in L[u]. Assume (∀x ∈ y)(∃z)(φ) holds in (L[∞],L(∞)).

By Lemma 5.8.22, L(∞)(x,y) → x < y. For each x such that L(∞)(x,y), we can consider the < least u such that (∃z ∈ L[u])(φ holds in (L[∞],L(∞))). This gives us an M^ definable function to which we can apply Lemma 5.8.1, and then take its strict sup, v, using Lemma 5.7.30 viii). By Lemma 5.8.19, set w ∈ L[v+1], where (∀v)(L(∞)(v,w) ↔ v ∈ L[u]). QED

DEFINITION 5.8.28. Let ZF\P be all axioms of ZF less Power Set, using Collection.

LEMMA 5.8.30. Every axiom of ZF\P with Collection holds in (L[∞],L(∞)).

Proof: From Lemmas 5.8.24, 5.8.25, 5.8.26, 5.8.29, 5.8.30. QED

Note that we have shown that all axioms of ZFC hold in (L[∞],L(∞)), with the exceptions of Power Set and Choice. In fact, we have verified Collection, which implies Replacement (in the presence of separation).

We now show that the power set axiom holds in (L[∞],L(∞)) using indiscernibility.

LEMMA 5.8.31. For all n ≥ 2, $L[c_n] \subseteq [0,c_{n+1})$.

Proof: Let n ≥ 2. Now $L[c_n]$ is internal, and in particular, bounded. By Lemma 5.7.30 v), let m > n be such that $L[c_n] \subseteq [0,c_m)$. We can view this as a true statement about c_n,c_m. By Lemma 5.7.30 ix), the statement is true of c_n,c_{n+1}. I.e., $L[c_n] \subseteq [0,c_{n+1})$. QED

DEFINITION 5.8.29. It is very convenient to define $x \subseteq^* y$ if and only if

$$x \in L[\infty] \wedge (\forall z \in L[\infty])(L(\infty)(z,x) \rightarrow L(\infty)(z,y)).$$

Also, $x \subseteq^{**} y$ if and only if

$$x \in L[\infty] \wedge (\forall z \in L[\infty])(L(\infty)(z,x) \rightarrow z \in L[y]).$$

LEMMA 5.8.32. Let $x \subseteq^{**} c_2$. Then $x < c_3$.

Proof: Suppose

$$1)\ (\exists x \geq c_3)(x \subseteq^{**} c_2).$$

By Lemma 5.7.30 ix), for every $n \geq 3$,

$$2)\ (\exists x \geq c_n)(x \subseteq^{**} c_2).$$

For each $n \geq 3$, let $J(n)$ be the $<$ least $x \geq c_n$ such that $x \subseteq^{**} c_2$.

Note that the $J(n)$, $n \geq 3$, are uniformly defined from c_2,c_n without parameters.

Fix $n \geq 3$. By Lemma 5.7.30 v), let $m > n$, and $J(n) < c_m$. By Lemma 5.7.30 ix), $J(n) < c_{n+1}$.

We have established that for all $n \geq 3$,

$$c_n \leq J(n) < c_{n+1} \wedge$$
$$\text{“}J(n) \subseteq L[c_2]\text{” holds in } (L[\infty],L(\infty)).$$

In particular, for all $n \geq 3$, $J(n) < J(n+1)$.

Let $y \in L[c_2]$. By Lemma 5.8.32, $y < c_3$. By Lemma 5.7.30 ix),

$$L(\infty)(y,J(4)) \leftrightarrow L(\infty)(y,J(5)).$$

This is because $J(4),J(5)$ are defined the same way from c_2,c_4 and from c_2,c_5, respectively, without parameters. I.e.,

$$3)\ (\forall y \in L[c_2])(L(\infty)(y,J(4)) \leftrightarrow L(\infty)(y,J(5))).$$

By the construction of J, we have

$$4)\ J(4) \subseteq^{**} c_2.$$
$$J(5) \subseteq^{**} c_2.$$
$$(\forall y \in L[\infty])(L(\infty)(z,J(4)) \rightarrow y \in L[c_2]).$$
$$(\forall y \in L[\infty])(L(\infty)(z,J(5)) \rightarrow y \in L[c_2]).$$

By 3),4), and extensionality in $(L[\infty],L(\infty))$, we have $J(4) = J(5)$. This contradicts $J(4) < J(5)$.

We have thus refuted 1). Hence

$$(\forall x)(x \subseteq^{**} c_2 \rightarrow x < c_3).$$

QED

LEMMA 5.8.33. Let $n \geq 2$ and $x \subseteq^{**} c_n$. Then $x < c_{n+1}$.

Proof: By Lemmas 5.8.32 and 5.7.30 ix). QED

LEMMA 5.8.34. Power Set holds in $(L[\infty],L(\infty))$.

Proof: Let $x \in L[\infty]$. By Lemma 5.7.30 v), let $x \in L[c_n]$, $n \geq 2$. Let $y \subseteq^* x$. Then $y \subseteq^{**} c_n$. By Lemma 5.8.33, $y < c_{n+1}$.

By Lemma 5.7.30 v), let $y \in L[c_m]$, $m \geq n+2$. By Lemma 5.7.30 ix), $y \in L[c_{n+2}]$. We have thus shown that for all y,

$$1)\ y \subseteq^* x \rightarrow y \in L[c_{n+2}].$$

Clearly $\{y \in L[c_{n+2}]: y \subseteq^* x\}$ is definable in $(L[c_{n+2}],L(c_{n+2}))$. Hence by Lemma 5.8.19, there exists $z \in L[c_{n+2}+1]$ such that

$$2)\ (\forall y)(y \subseteq^* x \leftrightarrow (L(c_{n+2}+1)(y,z))).$$

It follows that in $(L[\infty],L(\infty))$, z is the power set of x, using Lemma 5.8.21 (sharp extensions). Since $x \in L[\infty]$ is arbitrary, power set holds in $(L[\infty],L(\infty))$. QED

LEMMA 5.8.35. ZF holds in $(L[\infty],L(\infty))$. All sentences in $TR(\Pi^0_1,L)$ hold in $(L[\infty],L(\infty))$.

Proof: The first claim follows from Lemmas 5.8.30 and 5.8.34. For the second claim, from the proof of Lemma 5.8.25, we see that the finite von Neumann ordinals of $(L[\infty],L(\infty))$ are in order preserving one-one correspondence with $\{x: x < \omega\}$. Therefore the $0,1,+,-,\bullet,\uparrow,\log$ of $(L[\infty],L(\infty))$ is isomorphic to the $0,1,+,-,\bullet,\uparrow,\log$ of M^, by M^ induction, given the one-one correspondence and the operations are all internal to M^. The second claim now follows from Lemma 5.7.30 iii). QED

LEMMA 5.8.36. There exists a countable model M^+ of ZF + $TR(\Pi^0_1,L)$, with distinguished elements $d_1,d_2,\ldots$, such that
i) The d's are strictly increasing ordinals in the sense of M^+, without an upper bound;

ii) Let $r \geq 1$, and $i_1,\ldots,i_{2r} \geq 1$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min. Let R be a 2r-ary relation M^+ definable without parameters. Let $\alpha_1,\ldots,\alpha_r \leq \min(d_{i\_1},\ldots,d_{i\_r})$. Then $R(d_{i\_1},\ldots,d_{i\_r},\alpha_1,\ldots,\alpha_r) \leftrightarrow R(d_{i\_r+1},\ldots,d_{i\_2r},\alpha_1,\ldots,\alpha_r)$.

Proof: Take M^+ to be $(L[\infty],L(\infty))$. By Lemma 5.8.35, we have $ZF + TR(\Pi^0{}_1,L)$ in M^+.

For all $n \geq 1$, take d_n to be the minimum ordinal of $(L[\infty],L(\infty))$ lying outside $L[c_{2n}]$. In fact, $d_n \in L[c_{2n}+1]$ is the set of all ordinals in $L[c_{2n}]$, in the sense of $(L[\infty],L(\infty))$.

Note that $d_n \geq c_{2n}$ by Lemma 5.8.22. Also, since d_n is defined without parameters from c_{2n}, we have $d_n < c_{2n+1}$. I.e., for all n, $c_{2n} \leq d_n < c_{2n+1}$. Hence claim i) holds.

Let R be a 2r-ary relation M^+ definable without parameters. Then R is a 2r-ary relation on $L[\infty]$ that is M^ definable without parameters. Let $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min. Let the min be j. Let $\alpha_1,\ldots,\alpha_r \leq d_j$, where the α's are ordinals in the sense of M^+. In particular, $\alpha_1,\ldots,\alpha_r$ are ordinals of $(L[\infty],L(\infty))$. It follows that $\alpha_1,\ldots,\alpha_r < c_{2j+1}$.

We claim that

$$1)\ R(d_{i\_1},\ldots,d_{i\_r},\alpha_1,\ldots,\alpha_r) \leftrightarrow R(d_{i\_r+1},\ldots,d_{i\_2r},\alpha_1,\ldots,\alpha_r)$$

holds in M^+. To see this, replace each d_{i_p} by its definition in M^ from c_{2i_p}. Then 1) can be viewed as an assertion in M^ involving the parameters

2) $c_{2i\_1},\ldots,c_{2i\_r}$ on the left.
$c_{2i\_r+1},\ldots c_{2i\_2r}$ on the right.
$\alpha_1,\ldots,\alpha_r \leq c_{2j+1}$.
$j = \min(i_1,\ldots,i_{2r})$.

We can treat c_{2j} as an additional parameter. So we have the parameters

3) $c_{2i\_1},\ldots,c_{2i\_r}$ on the left, without c_{2j}.
$c_{2i\_r+1},\ldots c_{2i\_2r}$ on the right, without c_{2j}.
$\alpha_1,\ldots,\alpha_r,c_{2j} \leq c_{2j+1}$.
$j = \min(i_1,\ldots,i_{2r})$.

The 2j must occupy the same positions in $i_1,...,i_r$ as they do in $i_{r+1},...,i_{2r}$. Therefore, in 3), the remaining c's on the left have the same order type as the remaining c's on the right. But they do not necessarily have the same min. So we can insert a dummy variable at the end for c_{2j+1}. Thus we have

4) $c_{2i\_1},...,c_{2i\_r},c_{2j+1}$ on the left, without c_{2j}.
$c_{2i\_r+1},...c_{2i\_2r},c_{2j+1}$ on the right, without c_{2j}.
$\alpha_1,...,\alpha_r,c_{2j} \leq c_{2j+1}$.
$j = \min(i_1,...,i_{2r})$.

We now see that the equivalence holds because of Lemma 5.7.30 ix). QED

LEMMA 5.8.37. There exists a countable model M^+ of ZFC + V = L + TR($\Pi^0{}_1$,L), with distinguished elements $d_1,d_2,...$, such that
i) The d's are strictly increasing ordinals in the sense of M^+, without an upper bound;
ii) Let $r \geq 1$, and $i_1,...,i_{2r} \geq 1$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and min. Let R be a 2r-ary relation M^+ definable without parameters. Let $\alpha_1,...,\alpha_r \leq \min(d_{i\_1},...,d_{i\_r})$. Then $R(d_{i\_1},...,d_{i\_r},\alpha_1,...,\alpha_r) \leftrightarrow R(d_{i\_r+1},...,d_{i\_2r},\alpha_1,...,\alpha_r)$.

Proof: We could have proved the stronger form of Lemma 5.8.36, with ZFC + V = L instead of ZF. However, this would require a bit more than the usual hand waving with regards to internalized constructibility. So we have choose to wait until we have Lemma 5.8.36, with its honest to goodness model of ZF.

Start with the structure given by Lemma 5.8.36. Take the usual inner model of L. Ordinals are preserved. So we take the same d's, and i) is immediate. We still have TR($\Pi^0{}_1$,L), and since this inner model is definable without parameters, we preserve ii). QED