CHAPTER 6.
FURTHER RESULTS

6.1. Propositions D-H.

Our treatment of Propositions A,B,C culminated with Theorems 5.9.9, 5.9.11, and 5.9.12 at the end of Chapter 5.

In this section, we consider five Propositions D-H that have the same metamathematical properties as Propositions A,B,C. We will also consider some variants of Propositions D-H that do not share these properties, or whose status is left open.

Recall the main theorems of Chapter 5 (in section 5.9), which are Theorems 5.9.9, 5.9.11, and 5.9.12. Examination of the proofs of these three Theorems reveal that Theorem 5.9.11 with 1-Con(SMAH) is the key. If ACA' proves the equivalence of a statement with 1-Con(SMAH) then all of the other properties provided by these three Theorems quickly follow.

Accordingly, we establish these same three Theorems for Propositions D-H by showing that they are also each equivalent to 1-Con(SMAH) over ACA'.

We begin with Proposition D (see below), which is a sharpening of Proposition B. Proposition D immediately implies Propositions A-C over RCA_0.

Note that Propositions A-C are based on ELG. Examination of the proof of Proposition B in Chapter 4 shows that we can separately weaken the conditions on f,g in different ways. Also, we can place an inclusion condition on the starting set A_1. As usual, we use | | for the sup norm, or max. This results in Proposition D below.

DEFINITION 6.1.1. We say that f is linearly bounded if and only if f ∈ MF, and there exists d such that for all x ∈ dom(f),

$$f(x) \leq d|x|.$$

We let LB be the set of all linearly bounded f.

DEFINITION 6.1.2. We say that g is expansive if and only if $g \in MF$, and there exists $c > 1$ such that for all but finitely many $x \in dom(f)$,

$$c|x| \leq g(x)$$

We let EXPN be the set of all expansive g.

Recall the definitions of MF, SD (Definition 1.1.2), and ELG, EVSD (Definitions 2.1, 2.2).

PROPOSITION D. Let $f \in LB \cap EVSD$, $g \in EXPN$, $E \subseteq N$ be infinite, and $n \geq 1$. There exist infinite $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that
i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii) $A_1 \cap fA_n = \emptyset$;
iii) $A_1 \subseteq E$.

Note that $ELG \subseteq LB \cap EVSD \cap EXPAN$, and so Proposition D immediately implies Proposition B.

Proposition D is the strongest Proposition that we prove in this book (from large cardinals).

Recall that Propositions A-C are official statements of BRT. More accurately, Proposition B is really an infinite collection of statements of BRT.

Proposition D not a statement (or statements) of BRT for two reasons.

a. There is no common set of functions used for f,g (asymmetry).
b. The set E is used as data, rather than just f,g.

Features a,b both suggest very natural expansions of BRT. Feature a suggests “mixed BRT”, where one uses several classes of functions instead of just one. One can go further and use several classes of sets as well.

Feature b in Proposition D suggests another very natural expansion of BRT. In BRT, we consider statements of the form

given functions there are sets such that
a given Boolean relation holds between the sets
and their images under the functions.

We can expand BRT with

given functions and sets there are sets such that
a given Boolean relation holds between the sets
and their images under the functions.

We will not pursue such expansions of BRT in this book.

We remark that feature b can be removed (in some contexts such as here) by introducing a new function h and asserting that $A_1 \subseteq hN$ (obviously $hN = rng(h)$).

We now prove Proposition D in $SMAH^+$ by adapting the proof of Proposition B in $SMAH^+$ given in section 4.2.

We fix f,g,E as given by Proposition D. Analogously to section 4.2, we let f be p-ary, g be q-ary. We fix an integer $b \geq 1$ such that for all $x \in N^p$ and $y \in N^q$,

i. if $|x|,|y| > b$ then

$$|x| < f(x) \leq b|x|.$$
$$(1 + 1/b)|y| \leq g(y).$$

ii. if $|x| \leq b$ then $f(x) \leq b^2$.

Note how our inequalities are weaker than those used in section 4.2.

We also fix $n \geq 1$ and a strongly p^{n-1}-Mahlo cardinal κ.

The first place in section 4.2 that needs to be modified is at Lemma 4.2.2. Here we must use the given infinite set $E \subseteq N$.

LEMMA 4.2.2′. There exist infinite sets $E \supseteq E_0 \supseteq E_1 \supseteq \ldots$ indexed by N, such that for all $i \geq 0$, $\varphi \in AF(L)$, $lth(\varphi) \leq i$, and increasing partial $h_1,h_2:V(L) \rightarrow N$ adequate for φ with $rng(h_1),rng(h_2) \subseteq E_i$, we have $Sat(M,\varphi,h_1) \leftrightarrow Sat(M,\varphi,h_2)$.

Proof: See the proof of Lemma 4.2.2. QED

Lemma 4.2.3 do not involve our inequalities i,ii, and therefore require no modification.

We need to sharpen Lemma 4.2.4 for later purposes, since we do not have an upper bound for g. We use the # notation that was introduced much later just before Lemma 4.2.16.

LEMMA 4.2.4′. Let $\varphi \in AS(L^*)$. $Sat(M^*,\varphi)$ if and only if $\varphi \in T$. $<^*$ is a linear ordering on N^*. Let $n \geq 0$, $t \in CT(L^*)$, $\#(t) \leq n$. Then $t < c_{n+1} \in T$.

Proof: For the first claim, see the proof of Lemma 4.2.4. For the last claim, let $i = lth(t < c_{n+1})$. The unique increasing bijection $h:V(L) \rightarrow E_i$ has $Val(M,t',h) < h(v_{n+1})$, where t' is the result of replacing each c_i by v_i, using the indiscernibility of E_i. Argue as before. QED

Lemmas 4.2.5 - 4.2.8 do not involve our inequalities i,ii, and therefore require no modification.

We sharpen Lemma 4.2.9 for later purposes, since we do not have an upper bound for g.

LEMMA 4.2.9′. These definitions of $<^{**}$, $+^{**}$, f^{**}, g^{**} are well defined. Let $t \in CT(L^{**})$, $\#(t) \leq \alpha$. Then $t <^{**} c_{\alpha+1}{}^{**}$.

Proof: Use Lemma 4.2.4′ and the proof of Lemma 4.2.9. QED

Lemmas 4.2.10' - 4.2.14' do not involve our inequalities i,ii.

We need to weaken Lemma 4.2.15, in light of our inequalities i,ii.

LEMMA 4.2.15′. Let $x_1,\ldots,x_p,y_1,\ldots,y_q \in N^{**}$, where $|x_1,\ldots,x_p|,|y_1,\ldots,y_q| >^{**} b\hat{}$. Then

$$|x_1,\ldots,x_p| <^{**} f^{**}(x_1,\ldots,x_p) \leq^{**} b|x_1,\ldots,x_p|.$$
$$(1 + 1/b)|y_1,\ldots,y_q| \leq^{**} g^{**}(y_1,\ldots,y_q).$$

If $|x_1,\ldots,x_p| \leq^{**} b\hat{}$ then $f(x_1,\ldots,x_p) \leq^{**} b^2\hat{}$.

Proof: See the proof of Lemma 4.2.15. QED

We aim for a modification of the crucial well foundedness given by Lemma 4.2.19. This was stated using all elements of N^{**}. In other words, for all terms in $CT(L^{**})$. We cannot establish such a well foundedness result in the present setting for all terms in CT(L**). We have weakened the inequalities for f**,g** too much.

However, we can establish this well foundedness result for the restricted class of terms, CT(L**\g) consisting of all closed terms of L** in which g does not appear.

LEMMA 4.2.16′. Let t ∈ CT(L**). #(t) = -1 ↔ Val(M**,t) is standard. Suppose #(t) = c_α. Then c_α** ≤ Val(M**,t) <** $c_{\alpha+1}$**. Let s ∈ CT(L**\g). Suppose #(s) = c_α. There exists a positive integer d such that c_α** ≤** Val(M**,s) <** dc_α** <** $c_{\alpha+1}$**.

Proof: For the equivalence in the first claim, see the proof of Lemma 4.2.16. For the remaining claims, use induction on s,t, Lemmas 4.2.4′, 4.2.9′, 4.2.15′, and the proof of Lemma 4.2.16. QED

Lemmas 4.2.17, 4.2.18 do not involve our inequalities i,ii, and therefore require no modification.

DEFINITION 6.1.3. It is convenient to write VCT(L**\g) for the set of values of terms in CT(L**\g).

DEFINITION 6.1.4. Let s be a rational number. We write $<_s$**′ for the relation on VCT(L**\g) given by x $<_s$** y ↔ sx <** y.

LEMMA 4.2.19'. Let s be a rational number > 1. There exists k ≥ 1 such that for all x_1 $<_s$**' x_2 $<_s$**' ... $<_s$**' x_k, we have 2x_1 <**' x_k.

Proof: See the proof of Lemma 4.2.19. QED

Lemma 4.2.20 has to be weakened as follows.

LEMMA 4.2.20′. Let s be a rational number > 1. The relation $<_s$**′ on VCT(L**\g) is transitive, irreflexive, and well founded.

Proof: We adapt the proof of Lemma 4.2.20 with the following modification. In the fourth paragraph, d ∈ N\{0} is fixed such that Val(M**,t) <** dc_α**, using Lemma 4.2.16. Here we use Lemma 4.2.16' under the assumption that t ∈ VCT(L**\g). QED

DEFINITION 6.1.5. Let s = 1 + 1/2b for using Lemma 4.2.20'.

LEMMA 4.2.21′. There is a unique set W such that W = {x ∈ VCT(L**\g) ∩ nst(M**): x ∉ g**W}. For all $\alpha < \kappa$, c_α** ∉ rng(f**),rng(g**). In particular, each c_α** ∈ W.

Proof: Note that g**:NST(M**)^q → NST(M**), but g**:(VCT(L**\g) ∩ nst(M**))^q → VCT(L**\g) ∩ nst(M**) may be false. So we regard g** as a partial function from (VCTM(L**\g) ∩ nst(M**))^q into VCT(L**\g) ∩ nst(M*). Note that g** is strictly dominating from nst(M**) into nst(M**), in the sense of $<_s$**, by 4.2.15'. Since $<_s$** is well founded on VCT(L**\g) ∩ nst(M**), we can apply the Complementation Theorem for Well Founded Relations, proved in section 1.3 to obtain the first claim.

For the second claim, write c_α** = f**(x_1,...,x_p). By Lemma 4.2.15′, each x_i <** c_α**. By Lemma 4.2.18, f**(x_1,...,x_p) <** c_α**. This is a contradiction. The same argument applies to g**.

The third claim follows immediately from the second claim. QED

Lemma 4.2.22 - Theorem 4.2.26, Corollary 4.2.27, go through using the present W ⊆ VCT(L**\g) ∩ nst(M**), instead of the W ⊆ nst(M**) in section 4.2. We have shown the following.

THEOREM 6.1.1. Proposition D is provable in SMAH$^+$. For fixed arity of f and fixed n ≥ 1, Proposition D is provable in SMAH.

We now adapt section 4.4 to Proposition D. We redefine the p,q,b-structures, p,q,b;r-structures, p,q,b;n,r-special structures, p,q,b;r-types, p,q,b;n,r-special types, to take into account the weaker inequalities now placed on f,g. Specifically, clauses 4,5 in the definition of p,q,b-structure should now read

4′. f* obeys the above two inequalities for membership in LB(p,b) ∩ EVSD(p,b) given above right after we introduced Proposition D, internally in M*.
5′. g* obeys the above two inequalities for membership in EXPN(q,b), given above right after we introduced Proposition D, internally in M*.

These modified notions are written with ′.

The entire development of section 4.4 goes through without modification until we arrive at Theorem 4.4.11.

THEOREM 4.4.11′. Proposition D is provable in ACA′ + 1-Con(MAH).

Proof: We argue in ACA′ + 1-Con(MAH). Let $p,q,b,n \geq 1$, and $f \in LB(p,b) \cap EVSD(p,b)$, $g \in EXPN(q,b)$. Let r be given by Lemma 4.4.10′. By Ramsey's theorem for 2r-tuples in ACA′, we can find a p,q,b;r-structure′ M = $(N,0,1,<,+,f,g,c_0,c_1,\ldots)$, where $c_0,c_1,\ldots \in E$. Let τ be its p,q,b;r-type′. By Lemma 4.4.10′, τ is a p,q,b,n,r-special′ type. By Lemma 4.4.2, M is a p,q,b;r;n-special' structure. Let $D_1 \subseteq \ldots \subseteq D_n \subseteq N$, where $D_1 \subseteq \{c_0,c_1,\ldots\} \subseteq E$, and each $fD_i \subseteq D_{i+1} \cup. gD_{i+1}$, and $D_1 \cap fD_n = \emptyset$. This is Proposition D, thus concluding the proof. QED

THEOREM 6.1.2. ACA′ proves the equivalence of Proposition D and 1-Con(MAH), 1-Con(SMAH).

Proof: This is immediate from Theorems 4.4.11′, 5.9.11, and that Proposition D immediately implies Proposition B. QED

Recall that Proposition D is the strongest Proposition that we prove in this book (using large cardinals).

There are some natural variants of Proposition D, some of which are provable in RCA_0, and some of which are refutable.

PROPOSITION D[1]. Let $f,g \in EVSD$, $E \subseteq N$ be infinite, and $n \geq 1$. There exist infinite $A_1 \subseteq \ldots \subseteq A_n \subseteq N$ such that
i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii) $A_1 \cap fA_n = \emptyset$;
iii) $A_1 \subseteq E$.

Proposition D[1] is refutable in RCA_0. In fact, in section 6.3, we refute the following in RCA_0.

PROPOSITION α. For all $f,g \in SD \cap BAF$ there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

Note Proposition α follows immediately from Proposition D[1], even without E. This is because from the former, we get

$$A \cup. fA \subseteq B \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC$$
$$B \subseteq C$$
$$A \cup. fA \subseteq C \cup. gB.$$

Therefore Proposition D[1] is refutable in RCA_0 even if we remove E.

However, we can use EVSD if we drop the inclusions on the A's.

PROPOSITION D[2]. Let $f,g \in$ EVSD, $E \subseteq N$ be infinite, and $n \geq 1$. There exist infinite sets $A_1,...,A_n \subseteq N$ such that
i) for all $1 \leq i < n$, $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$;
ii) for all $1 \leq i \leq n$, $A_1 \cap fA_n = \emptyset$;
iii) $A_1 \subseteq E$.

The weakness in Proposition D[2] stems from the fact that we drop the tower condition, and use the same subscript twice on the right sides, and have no tower.

THEOREM 6.1.3. Proposition D[2] is provable in RCA_0.

Proof: Let f,g,E,n be as given. Let $t >> n \geq 1$. By a straightforward combinatorial argument, for all $t \geq 1$, we can find an infinite $E' \subseteq E$ such that

a. f,g are strictly dominating on the elements of their respective domains whose sup norm is at least min(E').
b. the values of all terms in f,g and elements of E', using at most t applications of functions, and at least one application of a function, lie outside E'.

We now inductively define $A_1,...,A_n$. Set $A_1 = E'$. Suppose $A_1,...,A_i$ have been defined for $1 \leq i < n$, where each A_j is an infinite subset of $[\min(E'),\infty)$. Set A_{i+1} to be the unique subset of fA_i such that $fA_i \subseteq A_{i+1} \cup. gA_{i+1}$. This unique A_{i+1} exists by i) above and Lemma 3.3.3. Also A_{i+1} is infinite since fA_i is infinite (using a) above).

It is clear by the construction of the A's, that all elements of the fA_i and gA_i meet the criterion in b) above for $t = n+1$, so that their values lie outside $E' = A_1$. This establishes Proposition D[2] in RCA_0. QED

Continuing with our use of EVSD, it is natural to consider the following.

PROPOSITION D[3]. Let $f,g \in$ EVSD and $n \geq 1$. There exist infinite sets $A_1,\ldots,A_n \subseteq N$ such that
i) for all $1 \leq i < j,k \leq n$, $fA_i \subseteq A_j \cup. gA_k$;
ii) $A_1 \cap fA_n = \emptyset$.

However, Proposition α is an obvious consequence of Proposition D[3] even for the case $n = 3$. So Proposition D[3] is refutable in RCA_0.

PROPOSITION D[4]. Let $f,g \in$ EVSD, $E \subseteq N$ be infinite, and $n \geq 1$. There exist infinite sets $A_1,\ldots,A_n \subseteq N$ such that
i) for all $1 \leq i < j,k \leq n$, $fA_i \subseteq A_j \cup. gA_k$;
ii) $A_1 \subseteq E$.

THEOREM 6.1.4. Proposition D[4] is provable in RCA_0.

Proof: Let f,g,E be as given. Let m be such that f,g are strictly dominating on $[m,\infty)$. Let B be unique such that $B \subseteq [m,\infty) \subseteq B \cup. gB$. Set $A_1 = E \cap [m,\infty)$, $A_2 = \ldots = A_n = B$. QED

PROPOSITION D[5]. Let $f,g \in$ EVSD (ELG, ELG $\cap$ SD $\cap$ BAF), $E \subseteq N$ be infinite, and $n \geq 1$. There exist $A_1,\ldots,A_n \subseteq N$ such that
i) for all $1 \leq i < j,k \leq n$, $fA_i \subseteq A_j \cup. gA_k$;
ii) for all $1 \leq i \leq n$, $A_i \cap E$ is infinite.

We do not know the status of Proposition D[5], other than it follows immediately from Proposition D.

We now present the remaining Propositions E,F that have the same metamathematical properties as Propositions A,B,C,D. These two propositions use ELG $\cap$ SD $\cap$ BAF.

DEFINITION 6.1.6. The powers of 2 are the integers 1,2,4,8,... . For $E \subseteq N$, we write $2^{(E)}$ for $\{2^n: n \in E\}$.

PROPOSITION E. For all $f,g \in$ ELG $\cap$ SD $\cap$ BAF there exist $A \subseteq B \subseteq C \subseteq N$, each containing infinitely many powers of 2, such that

$$fA \subseteq B \cup. gB$$
$$fB \subseteq C \cup. gC.$$

PROPOSITION F. For all $f,g \in ELG \cap SD \cap BAF$ there exist $A \subseteq B \subseteq C \subseteq N$, each containing infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC.$$

PROPOSITION G. For all $f,g \in ELG \cap SD \cap BAF$ there exist $A,B,C \subseteq N$, whose intersection contains infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC.$$

PROPOSITION H. For all $f,g \in ELG \cap SD \cap BAF$ there exist $A,B,C \subseteq N$, where $A \cap B$ contains infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC.$$

Note that Propositions E-H are statements in BRT, where the BRT setting consists of "subsets of N with infinitely many powers of 2", and $ELG \cap SD \cap BAF$. Propositions E,F,G immediately follow from Proposition D, using $E = 2^{(N)}$.

LEMMA 6.1.5. The following is provable in RCA_0. $D \rightarrow E \rightarrow F \rightarrow G \rightarrow H$.

Proof: For $D \rightarrow E$, let $E = 2^{(N)}$. For $E \rightarrow F$, use the derivation

$$fA \subseteq B \cup. gB$$
$$fB \subseteq C \cup gC$$
$$B \subseteq C$$
$$C \cap gB = \emptyset$$
$$fA \subseteq C \cup. gB.$$

$F \rightarrow G \rightarrow H$ is immediate. QED

We also consider two additional variants.

PROPOSITION E[1]. For all $f,g \in ELG \cap SD \cap BAF$ there exist $A,B,C \subseteq N$, whose intersection contains infinitely many powers of 2, such that

$$fA \subseteq B \cup. gB$$
$$fB \subseteq C \cup. gC.$$

PROPOSITION G[1]. For all f,g ∈ ELG ∩ SD ∩ BAF there exist A,B,C ⊆ N, each containing infinitely many powers of 2, such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC.$$

THEOREM 6.1.6. Proposition E[1] is provable in RCA_0.

Proof: Let f,g,E be as given. We follow the proof of Lemma 3.12.7. In the proof of Theorem 3.2.5, we can arrange that A ⊆ E. So in the proof of Lemma 3.12.7, we can assume that A ⊆ E. We also have A ⊆ B, A ⊆ C. QED

We do not know the status of Proposition G[1], even if we use ELG instead of ELG ∩ SD ∩ BAF. Obviously, this follows from Proposition D with $E = 2^{(N)}$.

Until Theorem 6.1.10, we work in RCA_0 and assume Proposition H.

LEMMA 6.1.7. For all f,g ∈ ELG ∩ SD ∩ BAF there exist infinite A,B,C ⊆ N such that

$$fA \subseteq C \cup. gB$$
$$fB \subseteq C \cup. gC$$
$$A \subseteq B,2^{(N)}.$$

Proof: Let f,g be as given. Let A,B,C be given by Proposition G. Replace A by $A \cap B \cap 2^{(N)}$, which is infinite. QED

LEMMA 6.1.8. The function f:N → N given by f(n) = 1 if n is a power of 2; 0 otherwise, lies in BAF.

Proof: Note that n is a power of 2 if and only if $n = 2^{\log(n)}$. QED

LEMMA 5.1.7'. Let f,g ∈ ELG ∩ SD ∩ BAF. There exist f′,g′ ∈ ELG ∩ SD ∩ BAF such that the following holds. Let S ⊆ N.
i) $g'S = g(S^*) \cup 12S+2 \cup (f(S^*) \cap 2^{(N+2)})$.
ii) $f'S = f(S^*) \cup g'S \cup 12f(S^*)+2 \cup 2S^*+1 \cup 3S^*+1$.

Proof: Let f,g ∈ ELG ∩ SD ∩ BAF, where $f:N^p \to N$ and $g:N^q \to N$. We define $g':N^{q+p} \to N$ as follows. Let $x_1,\ldots,x_q,y_1,\ldots,y_p \in N$.

case 1. $x_1,\ldots,x_q > y_1,\ldots,y_p$. Set $g'(x_1,\ldots,x_q,y_1,\ldots,y_p) = g(x_1,\ldots,x_q)$.

case 2. $y_1,\ldots,y_p > x_1,\ldots,x_q$ and $f(y_1,\ldots,y_p) \in 2^{(N+2)}$. Set $g'(x_1,\ldots,x_q,y_1,\ldots,y_p) = f(y_1,\ldots,y_p)$.

case 3. Otherwise. Set $g'(x_1,\ldots,x_q,y_1,\ldots,y_p) = 12|x_1,\ldots,x_q,y_1,\ldots,y_p|+2$.

We define $f':N^{5p+q+p} \to N$ as follows. Let $x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p \in N$.

case a. $|y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{2p+1},\ldots,x_{3p}| = |x_{3p+1},\ldots,x_{4p}| = |x_{4p+1},\ldots,x_{5p}|$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p) = g'(y_1,\ldots,y_q,z_1,\ldots,z_p)$.

case b. $|y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{2p+1},\ldots,x_{3p}| = |x_{3p+1},\ldots,x_{4p}| < \min(x_{4p+1},\ldots,x_{5p})$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p) = f(x_{4p+1},\ldots,x_{5p})$.

case c. $|y_1,\ldots,y_{q+1},z_1,\ldots,z_p| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{2p+1},\ldots,x_{3p}| = |x_{4p+1},\ldots,x_{5p}| < \min(x_{3p+1},\ldots,x_{4p}|$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p) = 12f(x_{3p+1},\ldots,x_{4p})+2$.

case d. $|y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_1,\ldots,x_p| = |x_{p+1},\ldots,x_{2p}| = |x_{3p+1},\ldots,x_{4p}| = |x_{4p+1},\ldots,x_{5p}| < \min(x_{2p+1},\ldots,x_{3p})$. Set $f'(x_1,\ldots,x_{3p},y_1,\ldots,y_q,z_1,\ldots,z_p| = 2|x_{2p+1},\ldots,x_{3p}|+1$.

case e. $|y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_1,\ldots,x_p| = |x_{2p+1},\ldots,x_{3p}| = |x_{3p+1},\ldots,x_{4p}| = |x_{4p+1},\ldots,x_{5p}| < \min(x_{2p+1},\ldots,x_{3p}|$. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p) = 3|x_{p+1},\ldots,x_{2p}|+1$.

case f. Otherwise. Set $f'(x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p) = 2|x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p|+1$.

Note that in case 1, $|x_1,\ldots,x_q,y_1,\ldots,y_p| = |x_1,\ldots,x_q|$, and in case 2, $|x_1,\ldots,x_q,y_1,\ldots,y_p| = |y_1,\ldots,y_p|$. Also note that in cases a)-e),

$$|x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p| = |y_1,\ldots,y_q,z_1,\ldots,z_p|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_{4p+1},\ldots,x_{5p}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_{3p+1},\ldots,x_{4p}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_{2p+1},\ldots,x_{3p}|$$
$$|x_1,\ldots,x_{5p},y_1,\ldots,y_q,z_1,\ldots,z_p| = |x_{p+1},\ldots,x_{2p}|$$

respectively. Hence $f',g' \in ELG \cap SD \cap BAF$.

Let S ⊆ N. From S, case 1 produces exactly g(S*). Case 2 produces exactly f(S*) ∩ $2^{(N+2)}$. Case 3 produces exactly 12S+2. This establishes i).

Case a) produces exactly g'S. Case b) produces exactly f(S*). Case c) produces exactly 12f(S*)+2. Case d) produces exactly 2S*+1. Case e produces exactly 3S*+1.

Case f) produces exactly 2S*+1 since 2min(S)+1 is not produced. This is because 2min(S)+1 is produced from case f) if and only if all of the arguments are min(S), which can only happen under case a). This establishes ii). QED

LEMMA 6.1.9. 12E+2, 6E, 2E+1 ∪ 3E+1, $2^{(N+2)}$ are pairwise disjoint, with the sole exception of 2E+1 ∪ 3E+1 and $2^{(N+2)}$.

Proof: Obviously, 12E+2, 6E, 2E+1 ∪ 3E+1 are pairwise disjoint by divisibility considerations. Also 12n+2 = 2m → 6n+1 = 2^{m-1}, which is impossible for m ≥ 3. QED

LEMMA 5.1.8'. Let f,g ∈ ELG ∩ SD ∩ BAF and rng(g) ⊆ 6N. There exist infinite A ⊆ B ⊆ C ⊆ N\{0} such that

i) fA ∩ 6N ⊆ B ∪ gB;
ii) fB ∩ 6N ⊆ C ∪ gC;
iii) fA ∩ 2N+1 ⊆ B;
iv) fA ∩ 3N+1\$2^{(N+2)}$ ⊆ B;
v) fB ∩ 2N+1 ⊆ C;
vi) fB ∩ 3N+1\$2^{(N+2)}$ ⊆ C;
vii) C ∩ gC = ∅;
viii) A ∩ fB = ∅.

Proof: Let f,g be as given. Let f',g' be given by Lemma 5.1.7'. Let A,B,C ⊆ N be given by Lemma 6.1.7 for f',g'. Then A,B,C are infinite, and

f'A ⊆ C ∪. g'B
f'B ⊆ C ∪. g'C
A ⊆ B,$2^{(N)}$.

Since we can shrink A to any infinite subset, we will assume that A ⊆ $2^{(N+2)}$.

Let n ∈ B. Then 12n+2 ∈ g'B ∩ f'B, and so 12n+2 ∈ C ∪ g'C. Now 12n+2 ∉ C by C ∩ g'B = ∅. Hence 12n+2 ∈ g'C. Therefore 12n+2 ∈ 12C+2. Hence n ∈ C. So we have established that A ⊆ B ⊆ C.

We now verify all of the required conditions i)-viii) above using the three sets A*,B*,C*.

Firstly note that $A^* \subseteq B^* \subseteq C^* \subseteq N\backslash\{0\}$. To see this, first observe that $\min(A) \geq \min(B) \geq \min(C)$. Now let $n \in A^*$. Then $n \in B \wedge n > \min(A) \geq \min(B)$. Hence $n \in B^*$. Thus $A^* \subseteq B^*$. The same argument establishes $B^* \subseteq C^*$.

We now claim that $A^* \cap f(B^*) = \emptyset$. Let $n \in A^*$, $n \in f(B^*)$. Then $n \in f(B^*) \cap 2^{(N+2)}$, $n \in g'B$, $n \in C$. This is a contradiction.

Next we claim that $C^* \cap g(C^*) = \emptyset$. This follows from $C \subseteq C^*$, $g(C^*) \subseteq g'C$, and $C \cap g'C = \emptyset$.

Now we claim that $f(A^*) \cap 6N \subseteq B^* \cup g(B^*)$. To see this, let $n \in f(A^*) \cap 6N$. Then $n \in f'A$, $n \in C \cup g'B$.

case 1. $n \in C$. Now $12n+2 \in g'C$ and $12n+2 \in 12f(A^*)+2 \subseteq f'A$. Since $C \cap g'C = \emptyset$, we have $12n+2 \notin C$. Also $12n+2 \in C \cup g'B$. Hence $12n+2 \in g'B$. Therefore $12n+2 \in 12B+2$, and so $n \in B$. Since $n \in f(A^*)$ and f is strictly dominating, we have $n > \min(A) \geq \min(B)$. Hence $n \in B^*$.

case 2. $n \in g'B$. Since $n \in 6N$, $n \in g(B^*)$. This establishes the claim.

Next we claim that $f(B^*) \cap 6N \subseteq C^* \cup g(C^*)$. To see this, let $n \in f(B^*) \cap 6N$. Then $n \in f'B$. Hence $n \in C \cup g'C$.

case 1'. $n \in C$. Since $n \in f(B^*)$ and f is strictly dominating, we have $n > \min(B) \geq \min(C)$. Hence $n \in C^*$.

case 2'. $n \in g'C$. Since $n \in 6N$, we have $n \in g(C^*)$. This establishes the claim.

Now we claim that $f(A^*) \cap 2N+1$, $f(A^*) \cap 3N+1\backslash 2^{(N+2)} \subseteq B^*$. To see this, let $n \in f(A^*)$, $n \in 2N+1 \cup 3N+1$, $n \notin 2^{(N+2)}$. Note that $n \notin rng(g')$. Also, $n \in f'A$, $n \in C \cup g'B$. Hence $n \in C$, $12n+2 \in g'C$, $12n+2 \notin C$. Now $12n+2 \in 12f(A^*)+2 \subseteq f'A \subseteq C \cup g'B$, $12n+2 \in g'B$, $n \in B$. Since f is strictly dominating, $n > \min(A) \geq \min(B)$, and so $n \in B^*$.

Finally we claim that $f(B^*) \cap 2N+1$, $f(B^*) \cap 3n+1\backslash 2^{(N+2)} \subseteq C^*$. To see this, let $n \in f(B^*)$, $n \in 2N+1 \cup 3N+1$, $n \notin 2^{(N+2)}$. Note that $n \notin rng(g')$. Also, $n \in f'B$, $n \in C \cup g'C$. Hence $n \in C$, $12n+2 \in g'C$, $12n+2 \notin C$. Now $12n+2 \in 12f(B^*)+2 \subseteq f'B \subseteq$

$C \cup g'C$. Hence $12n+2 \in g'C$, $n \in C$. Since f is strictly dominating, $n > \min(B) \geq \min(C)$, and so $n \in C^*$. QED

The proof of 1-Con(SMAH) from Proposition C given in Chapter 5 is strictly modular, in that we can start with Lemma 5.1.8 instead of Proposition C.

Here we repeat the proof in Chapter 5 using Lemma 5.1.8' instead of Lemma 5.1.8. However, Lemma 5.1.8' is slightly weaker than Lemma 5.1.8, because of the weakened clauses iv) and vi), where we use $3N+1\backslash 2^{(N+2)}$ instead of $3N+1$.

So we need to identify the few places at which we use $3N+1$ and make sure that we can get away with $3N+1\backslash 2^{(N+2)}$ instead.

By examination of the proofs, we obtain the following series of slightly weakened Lemmas from the end of sections 5.1 - 5.5. Finally, we show that we obtain Lemma 5.6.20 without modification.

LEMMA 5.2.12'. Let $r \geq 3$ and $g \in ELG \cap SD \cap BAF$, where $rng(g) \subseteq 48N$. There exists $(D_1,\ldots,D_r)$ such that
i) $D_1 \subseteq \ldots \subseteq D_r \subseteq N\backslash\{0\}$;
ii) $|D_1| = r$ and D_r is finite;
iii) for all $x < y$ from D_1, $x\uparrow < y$;
iv) for all $1 \leq i \leq r-1$, $48\alpha(r,D_i;1,r) \subseteq D_{i+1} \cup gD_{i+1}$;
v) for all $1 \leq i \leq r-1$, $2\alpha(r,D_i;1,r)+1$, $3\alpha(r,D_i;1,r)+1\backslash 2^{(N+2)} \subseteq D_{i+1}$;
vi) $D_r \cap gD_r = \emptyset$;
vii) $D_1 \cap \alpha(r,D_2;2,r) = \emptyset$;
viii) Let $1 \leq i \leq \beta(2r)$, $x_1,\ldots,x_{2r} \in D_1$, $y_1,\ldots,y_r \in \alpha(r,D_2)$, where $(x_1,\ldots,x_r)$ and $(x_{r+1},\ldots,x_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(x_1,\ldots,x_r)$. Then
$t[i,2r](x_1,\ldots,x_r,y_1,\ldots,y_r) \in D_3 \leftrightarrow$
$t[i,2r](x_{r+1},\ldots,x_{2r},y_1,\ldots,y_r) \in D_3$.

LEMMA 5.3.18'. There exists a countable structure $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,\ldots)$ such that the following holds.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0_1,L)$;
ii) $E \subseteq A\backslash\{0\}$;
iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements in $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;
iv) Let $r,n \geq 1$, $t(v_1,\ldots,v_r)$ be a term of L, and $x_1,\ldots,x_r \leq c_n$. Then $t(x_1,\ldots,x_r) < c_{n+1}$;
v) $2\alpha(E;1,<\infty)+1$, $3\alpha(E;1,<\infty)+1\backslash 2^{(A+2)} \subseteq E$;

vi) Let $r \geq 1$, $a,b \in N$, and $\varphi(v_1,\ldots,v_r)$ be a quantifier free formula of L. There exist $d,e,f,g \in N\backslash\{0\}$ such that for all $x_1 \in \alpha(E;1,<\infty)$, $(\exists x_2,\ldots,x_r \in E)(x_2,\ldots,x_r \leq ax_1+b \wedge \varphi(x_1,\ldots,x_r)) \leftrightarrow dx_1+e \notin E \leftrightarrow fx_1+g \in E$;
vii) Let $r \geq 1$, $p \geq 2$, and $\varphi(v_1,\ldots,v_{2r})$ be a quantifier free formula of L. There exist $a,b,d,e \in N\backslash\{0\}$ such that the following holds. Let $n \geq 1$ and $x_1,\ldots,x_r \in \alpha(E;1,<\infty) \cap [0,c_n]$. Then
$(\exists y_1,\ldots,y_r \in E)(y_1,\ldots,y_r \leq \uparrow p(|x_1,\ldots,x_r|) \wedge$
$\varphi(x_1,\ldots,x_r,y_1,\ldots,y_r)) \leftrightarrow$
$a\text{CODE}(c_{n+1};x_1,\ldots,x_r)+b \notin E \leftrightarrow$
$d\text{CODE}(c_{n+1};x_1,\ldots,x_r)+e \in E$. Here CODE is as defined just before Lemma 5.3.11;
viii) Let $k,n,m \geq 1$, and $x_1,\ldots,x_k \leq c_n < c_m$, where $x_1,\ldots,x_k \in \alpha(E;1,<\infty)$. Then $\text{CODE}(c_m;x_1,\ldots,x_k) \in E$;
ix) Let $r \geq 1$ and $t(v_1,\ldots,v_{2r})$ be a term of L. Let $i_1,\ldots,i_{2r} \geq 1$ and $y_1,\ldots,y_r \in E$, where $(i_1,\ldots,i_r)$ and $(i_{r+1},\ldots,i_{2r})$ have the same order type and min, and $y_1,\ldots,y_r \leq \min(c_{i\_1},\ldots,c_{i\_r})$. Then
$t(c_{i\_1},\ldots,c_{i\_r},y_1,\ldots,y_r) \in E \leftrightarrow$
$t(c_{i\_r+1},\ldots,c_{i\_2r},y_1,\ldots,y_r) \in E$.

Lemma 5.4.12 uses $2\alpha(E;1,<\infty)+1$, $3\alpha(E;1,<\infty)+1 \subseteq E$. However, we only have $3\alpha(E;1,<\infty)+1\backslash 2^{(A+2)} \subseteq E$. So it suffices to augment the displayed derivation in Lemma 5.4.12 with the second derivation

$$t(x_1,\ldots,x_k) < c_{n+1}.$$
$$2c_{n+1}+t(x_1,\ldots,x_k)+3, 3c_{n+1}+t(x_1,\ldots,x_k)+2 \in \alpha(E;1,<\infty).$$
$$3(2c_{n+1}+t(x_1,\ldots,x_k)+2)+1,\ 2(3c_{n+1}+t(x_1,\ldots,x_k)+3)+1 \in E.$$
$$6c_{n+1}+3t(x_1,\ldots,x_k)+7,\ 6c_{n+1}+2t(x_1,\ldots,x_k)+7 \in E.$$
$$(6c_{n+1}+3t(x_1,\ldots,x_k)+7)-(6c_{n+1}+2t(x_1,\ldots,x_k)+7) =$$
$$t(x_1,\ldots,x_k) \in E-E.$$

provided we verify that

$$3(2c_{n+1}+t(x_1,\ldots,x_k))+1 \notin 2^{(A+2)} \vee 3(2c_{n+1}+t(x_1,\ldots,x_k)+2)+1 \notin 2^{(A+2)}.$$

This is evident, since any two powers of 2 that are ≥ 4 cannot differ by 6.

LEMMA 5.4.17'. There exists a countable structure $M = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,\ldots)$, and terms $t_1,t_2,\ldots$ of L, where for all i, t_i has variables among $v_1,\ldots,v_{i+8}$, such that the following holds.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0{}_1,L)$;

ii) $E \subseteq A\backslash\{0\}$;
iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements in $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;
iv) Let $r,n \geq 1$ and $t(v_1,...,v_r)$ be a term of L, and $x_1,...,x_r \leq c_n$. Then $t(x_1,...,x_r) < c_{n+1}$;
v) $2\alpha(E;1,<\infty)+1$, $3\alpha(E;1,<\infty)+1\backslash 2^{(A+2)} \subseteq E$;
vi) Let $k,n \geq 1$ and R be a c_n-definable k-ary relation. There exists $y_1,...,y_8 \in E \cap [0,c_{n+1}]$ such that $R = \{(x_1,...,x_k) \in E^k \cap [0,c_n]^k: t_k(x_1,...,x_k,y_1,...,y_8) \in E\}$;
vii) Let $r \geq 1$ and $\varphi(v_1,...,v_{2r})$ be a formula of L(E). Let $1 \leq i_1,...,i_{2r} < n$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and the same min. Let $y_1,...,y_r \in E$, $y_1,...,y_r \leq \min(c_{i\_1},...,c_{i\_r})$. Then $\varphi(c_{i\_1},...,c_{i\_r},y_1,...,y_r)^{c\_n} \leftrightarrow \varphi(c_{i\_r+1},...,c_{i\_2r},y_1,...,y_r)^{c\_n}$.

LEMMA 5.5.8'. There exists a countable structure $M^* = (A,<,0,1,+,-,\bullet,\uparrow,\log,E,c_1,c_2,...,X_1,X_2,...)$, where for all $i \geq 1$, X_i is the set of all i-ary relations on A that are c_n-definable for some $n \geq 1$; and terms $t_1,t_2,...$ of L, where for all i, t_i has variables among $x_1,...,x_{i+8}$, such that the following holds.
i) $(A,<,0,1,+,-,\bullet,\uparrow,\log)$ satisfies $TR(\Pi^0{}_1,L)$;
ii) $E \subseteq A\backslash\{0\}$;
iii) The c_n, $n \geq 1$, form a strictly increasing sequence of nonstandard elements of $E\backslash\alpha(E;2,<\infty)$ with no upper bound in A;
iv) For all $r,n \geq 1$, $\uparrow r(c_n) < c_{n+1}$;
v) $2\alpha(E;1,<\infty)+1$, $3\alpha(E;1,<\infty)+1\backslash 2^{(A+2)} \subseteq E$;
vi) Let $k,n \geq 1$ and R be a c_n-definable k-ary relation. There exists $y_1,...,y_8 \in E \cap [0,c_{n+1}]$ such that $R = \{(x_1,...,x_k) \in E^k \cap [0,c_n]^k: t_k(x_1,...,x_k,y_1,...,y_8) \in E\}$;
vii) Let $k \geq 1$, $m \geq 0$, and φ be an E formula of L*(E) in which R is not free, where all first order variables free in φ are among $x_1,...,x_{k+m+1}$. Then $x_{k+1},...,x_{k+m+1} \in E \rightarrow (\exists R)(\forall x_1,...,x_k \in E)(R(x_1,...,x_k) \leftrightarrow (x_1,...,x_k \leq x_{k+m+1} \wedge \varphi))$;
viii) Let $r \geq 1$, and $\varphi(x_1,...,x_{2r})$ be an E formula of L*(E) with no free second order variables. Let $1 \leq i_1,...,i_{2r}$, where $(i_1,...,i_r)$ and $(i_{r+1},...,i_{2r})$ have the same order type and the same min. Let $x_1,...,x_r \in E$, $x_1,...,x_r \leq \min(c_{i\_1},...,c_{i\_r})$. Then $\varphi(c_{i\_1},...,c_{i\_r},x_1,...,x_r) \leftrightarrow \varphi(c_{i\_r+1},...,c_{i\_2r},x_1,...,x_r)$.

Lemma 5.6.2 involves reproving a weak form of Lemma 5.4.12 using a related construction. Here $3\alpha(E;1,<\infty)+1 \subseteq E$ can also be replaced by $3\alpha(E;1,<\infty)+1\backslash 2^{(A+2)}$, also by the same method.

In the remainder of section 5.6, we do not use $3\alpha(E;1,<\infty)+1\backslash 2^{(A+2)} \subseteq E$. Hence we obtain Lemma 5.6.20. We have proved the following.

THEOREM 6.1.10. ACA' proves that each of Propositions A-H are equivalent to Con(SMAH).

Proof: We have completed the proof that ACA' proves Proposition H implies 1-Con(SMAH). The result follows by Lemmas 5.9.11 and 6.1.5. QED