6.3. A Refutation.

In Proposition A, can we replace ELG by the simpler and more basic SD? We refute this in a strong way. In particular, we refute Proposition C with ELG removed.

PROPOSITION α. For all $f,g \in SD \cap BAF$ there exist $A,B,C \in INF$ such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

We will even refute the following weaker Proposition.

PROPOSITION β. Let $f,g \in SD \cap BAF$. There exist $A,B,C \subseteq N$, $|A| \geq 4$, such that

$$A \cup. fA \subseteq C \cup. gB$$
$$A \cup. fB \subseteq C \cup. gC.$$

We assume Proposition β, and derive a contradiction.

We begin with a modification of Lemmas 5.1.6 and 5.1.7. Basically, these go through without any change in the proof, but we provide some additional details.

LEMMA 5.1.6'. Let $f,g \in SD \cap BAF$. There exist $f',g' \in SD \cap BAF$ such that the following holds.
i) $g'S = g(S^*) \cup 6S+2$;
ii) $f'S = f(S^*) \cup g'S \cup 6f(S^*)+2 \cup 2S^*+1 \cup 3S^*+1$.

Proof: In the proof of Lemma 5.1.6, f',g' are constructed explicitly from f,g. It is obvious that if $f,g \in SD \cap BAF$, then $f',g' \in SD \cap BAF$. The verification goes through without change. QED

LEMMA 5.1.7'. Let $f,g \in SD \cap BAF$ and $rng(g) \subseteq 6N$. There exist $A \subseteq B \subseteq C \subseteq N\backslash\{0\}$, $|A| \geq 3$, such that
i) $fA \cap 6N \subseteq B \cup gB$;
ii) $fB \cap 6N \subseteq C \cup gC$;
iii) $fA \cap 2N+1 \subseteq B$;
iv) $fA \cap 3N+1 \subseteq B$;
v) $fB \cap 2N+1 \subseteq C$;
vi) $fB \cap 3N+1 \subseteq C$;
vii) $C \cap gC = \emptyset$;
viii) $A \cap fB = \emptyset$;

Proof: In the proof Lemma 5.1.6, f',g' are constructed explicitly from f,g. Then A,B,C are used from Proposition

C, and it is verified that $A \subseteq B \subseteq C$, $A^* \subseteq B^* \subseteq C^* \subseteq N\backslash\{0\}$, and A^*,B^*,C^* obey i) - viii). Suppose $f,g \in SD \cap BAF$, $rng(g) \subseteq 6N$. Then $f',g' \in SD \cap BAF$, and we take A,B,C from Proposition β, $|A| \geq 4$. The same argument shows that $A \subseteq B \subseteq C$, $A^* \subseteq B^* \subseteq C^* \subseteq N\backslash\{0\}$, and A^*,B^*,C^* obey i) - viii). Obviously $|A^*| \geq 3$. QED

LEMMA 6.3.1. Suppose $n > m \wedge x > c \wedge 48n\uparrow-24m = 48x\uparrow-24c$. Then $n = x \wedge m = c$.

Proof: Let n,m,x,c be as given. Then

$$48n\uparrow-48x\uparrow = 24m-24c.$$
$$2(n\uparrow-x\uparrow) = m-c.$$
$$n \neq x \rightarrow \max(n,x)\uparrow \leq |2(n\uparrow-x\uparrow)| = |m-c| < \max(n,x).$$
$$n = x,\ m = c.$$

QED

Define $f:N^5 \rightarrow N$ as follows. Let $a,b,c,d,e \in N$.

case 1. $a = b = c \wedge |a,b,c,d,e| = e$. Define $f(a,b,c,d,e) = e+1$.

case 2. $a = b > c \wedge |a,b,c,d,e| = e$. Define $f(a,b,c,d,e) = e+2$.

case 3. $a = b < c \wedge |a,b,c,d,e| = e$. Define $f(a,b,c,d,e) = 48e\uparrow+12$.

case 4. $a < b = c \wedge |a,b,c,d,e| = e$. Define $f(a,b,c,d,e) = 48e\uparrow-24d$.

case 5. $a < b \wedge a = c \wedge |a,b,c,d,e| = e$. Define $f(a,b,c,d,e) = 48e\uparrow-24(d+1)$.

case 6. $a > b = c \wedge |a,b,c,d,e| = e$. Define $f(a,b,c,d,e) = 48e\uparrow-24(d+2)$.

case 7. otherwise. Define $f(a,b,c,d,e) = |a,b,c,d,e|+1$.

Define $g:N^5 \rightarrow 6N$ as follows. Let $n,t,m,r,s \in N$.

case 1. $n = t > m > r$, $s = 48n\uparrow-24m$. Define $g(n,t,m,r,s) = 48n\uparrow-24r$.

case 2. $n > t = m > r$, $s = 48n\uparrow-24m$. Define $g(n,t,m,r,s) = 48n\uparrow+12$.

case 3. otherwise. Define $g(n,t,m,r,s) = 48|n,t,m,r,s|+6$.

Note the modest use of t in the definition of g.

LEMMA 6.3.2. $f,g \in SD \cap BAF$. For all $S \subseteq N$, $S^*+1 \cup S^*+2 \cup \{48n{\uparrow}+12: n \in S^*\} \cup \{48n{\uparrow}-24(m+j): n,m \in S^* \wedge m \leq n \wedge j \leq 2\} \subseteq fS$. The outputs of cases 1-3 in the definition of g are pairwise disjoint.

Proof: Let $S \subseteq N$. At arguments from S, case 1 in the definition of f produces $S+1$; case 2 produces S^*+2, case 3 produces $48n{\uparrow}+12$, $n \in S^*$; case 4 produces the $48n{\uparrow}-24m$, $n,m \in S^*$, $m \leq n$; case 5 produces the $48n{\uparrow}-24(m+1)$, $n,m \in S^*$, $m \leq n$, and case 6 produces the $48n{\uparrow}-24(m+2)$, $n,m \in S^*$, $m \leq n$. (In cases 4-6, additional integers can be produced). Since $e > 0 \rightarrow 48e{\uparrow}-24(e+2) > e$, we see that $f \in SD \cap BAF$.

Note that if $n > m > r$ then $48n{\uparrow}-24r > 24n{\uparrow} > n$, and $48n{\uparrow}-24r > 48n{\uparrow}-24m$. Also, if $n > m > r$ then $48n{\uparrow}+12 > 48n{\uparrow}-24m,n$. Hence $g \in SD \cap BAF$.

The three cases in the definition of g yield integers congruent to 24,12,6 modulo 48, respectively. QED

We now apply Lemma 5.1.7' to f,g. Fix A,B,C according to Lemma 5.1.7'.

LEMMA 6.3.3. Let $n \in C$. There is at most one $m \in C$ such that $m < n \wedge 48n{\uparrow}-24m \in C$.

Proof: Let $m,m' \in C$, $m < m' < n$, $48n{\uparrow}-24m, 48n{\uparrow}-24m' \in C$. Then $g(n,n,m',m,48n{\uparrow}-24m') = 48n{\uparrow}-24m$. Hence $48n{\uparrow}-24m \in C \cap gC$, which contradicts Lemma 5.1.7' vii). QED

LEMMA 6.3.4. Let $n \in A^*$. Then $(\forall m \in C^*)(m < n \rightarrow 48n{\uparrow}-24m \notin C)$.

Proof: Let $n \in A^*$, $m \in C^*$, $m < n$, $48n{\uparrow}-24m \in C$. Then $g(n,m,m,\min(C),48n{\uparrow}-24m) = 48n{\uparrow}+12 \in gC$. By Lemma 5.1.7' vii), $48n{\uparrow}+12 \notin C$. By Lemma 6.3.2, $48n{\uparrow}+12 \in fA \cap 6N$. By Lemma 5.1.7' i), we have $48n{\uparrow}+12 \in B \cup gB$. Hence $48n{\uparrow}+12 \in gB$. Let $48n{\uparrow}+12 = g(a,t,b,c,d)$, $a,t,b,c,d \in B$. Then case 2 applies and $g(a,t,b,c,d) = 48a{\uparrow}+12$, $d = 48a{\uparrow}-24b$, $a > b > c$. Obviously $a = n$ and $b \in B^*$.

Thus we have $b < n$, and $48n\uparrow-24b \in B$, $b,n \in B$. Note that $m < n$, $48n\uparrow-24m \in C$, $m,n \in C$. By Lemma 6.3.3, $m = b < a = n$. Hence $m \in B^*$, $m < n$, $48n\uparrow-24m \in B$.

By Lemma 6.3.2, $m+1,m+2 \in fB$. By Lemma 5.1.7' viii), we have $m+1,m+2 \notin A$. In particular, $m+1,m+2 \neq n$. Since $m < n$, we have $m+2 < n$. Let $i \in \{1,2\}$ be such that $m+i$ is odd, $m+i < n$. By Lemma 5.1.7' v), $m+i \in C$.

By Lemma 6.3.3, $48n\uparrow-24(m+i) \notin C$. Note that $n,m \in B^*$, $m < n$, and so by Lemma 6.3.2, $48n\uparrow-24(m+i) \in fB$. By Lemma 5.1.7' ii), $48n\uparrow-24(m+i) \in C \cup gC$, $48n\uparrow-24(m+i) \in gC$. Let $48n\uparrow-24(m+i) = g(x,t,b,c,d)$, $x,t,b,c,d \in C$. Then case 1 applies and $g(x,t,b,c,d) = 48x\uparrow-24c$, $d = 48x\uparrow-24b$, $x > b > c$. By Lemma 6.3.1, $x = n \wedge m+i = c < b$. Hence $b < n$, $48n\uparrow-24b = d \in C$, $b,n \in C$. By Lemma 6.3.3, $b = m$. This contradicts $b > m+i$. QED

THEOREM 6.3.5. Proposition α is refutable in RCA_0. In fact, Proposition β is refutable in RCA_0.

Proof: Let $s,n \in A^*$, $s < n$. This is supported by $|A| \geq 3$. Hence $s \in C^*$. By Lemma 6.3.4, $48n\uparrow-24s \notin C$. By Lemma 6.3.2, $48n\uparrow-24s \in fA$. By Lemma 5.1.7' i), we have $48n\uparrow-24s \in B \cup gB$, $48n\uparrow-24s \in gB$. Let $48n\uparrow-24s = g(a,t,b,c,d)$, $a,t,b,c,d \in B$. Then case 1 applies, and $g(a,t,b,c,d) = 48a\uparrow-24c$, $a > b > c$, $d = 48a\uparrow-24b$. By Lemma 6.3.1, $a = n \wedge c = s$. Now $b < n \wedge 48n\uparrow-24b \in B$. Clearly $b \in B^* \subseteq C^*$. This contradicts Lemma 6.3.4. QED