A COMPLETE THEORY OF EVERYTHING: satisfiability in the universal domain

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October 10, 1999
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1. GENERAL REMARKS

Let LPC(=) be the usual language of predicate calculus with equality, and PC(=) be a standard system of axioms and rules and inference for predicate calculus with equality.

The completeness theorem determines those sentences that have models with certain domains. Here are five cases:

1. Nonempty sets.
2. N.
3. Infinite sets.
4. Fixed infinite set.
5. Extensions of unary predicates.

Very few basic principles are needed for these results. 4 needs the axiom of choice.

Note that 5 does not have the accepted clarity of meaning that 1-4 have. In fact, we will be considering two major distinct interpretations of 5. Nevertheless, the completeness theorem applies to 5, since its proof is so robust.

We call 5 the domain interpretation.

Here we take the view that LPC(=) is applicable to structures whose domain is too large to be a set. This is not just a matter of class theory versus set theory, although it can be interpreted as such, and this interpretation is discussed briefly at the end.

We apply LPC(=) to the largest domain of all - the domain W consisting of absolutely everything.

In order to make sense of this, we need the concept of arbitrary predicate of several variables on W.

This context is often viewed as a dangerous slippery slope, fraught with paradoxes, or at least, quite murky, subject to differing interpretations.

But the thrust of this work is that surprisingly minimal principles about predication on W are needed to establish which sentences of LPC(=) are satisfiable in domain W, and related results.

2. TWO NOTIONS OF PREDICATION AND INTERPRETATIONS

In the context of predication on W, identity of indiscernibles states the following:

for all x,y, x = y iff for all unary predicates P on W,
$$P(x) \leftrightarrow P(y).$$

This follows from singleton extensions principle:

for all x there exists a unary predicate P on W such that
$$\forall y(P(y) \leftrightarrow y = x).$$

In mathematical and set theoretic predication, the singleton extensions principle holds.

However, there is another concept of predication discussed by philosophers, for which the singleton extensions principle is doubtful.

This is the notion of predicates that are given without reference to any particular objects, but only involving concepts. Or one can take a linguistic tack - relations that can be defined in a language, where language is broadly interpreted, rather than in some particular already formalized language.

We will use the term "general predicate" for the first concept of predication, where the singleton extensions principle obviously holds.

And we will use the term "pure predicate" for the second concept of predication, where even the doctrine of identity of indiscernibles is problematic.

What is the relationship between these two concepts, of pure predicates and of general predicates? Obviously every extension of a pure predicate is the extension of a general predicate.

However, it does not seem that one can hope to characterize the extensions of pure predicates in terms of the extensions of general predicates.

One reasonable philosophical position is that the concept of pure predication on W is fundamental, whereas any concept of general predication on W is derived. Under this position, general predicates are defined as cross sections of pure predicates. We will not take this view here, but view pure and general predication as separate, but related, fundamental concepts.

In particular, we now have the Wp and Wg interpretations of LPC(=). The domain is always W and the constant symbols are always elements, but the relations and functions are pure and general, respectively. Functions are always thought of as univalent predicates.
A sentence in LPC(=) is called Wp (Wg) satisfiable if and only if it is true in some Wp (Wg) interpretation. This work deals with the determination of the Wp (Wg) satisfiable sentences of LPC(=).

3. DOMAIN INTERPRETATION SUBSUMED

Recall the domain interpretation. We distinguish between the pure and the general domain interpretation. We indicate how it is subsumed by the Wg (Wp) interpretations.

□ is satisfiable in the general (pure) domain interpretation is equivalent to the Wg (Wp) satisfiability of

$$(\square x)(P(x)) \;\square\; \square^{(P)}$$

where $\square^{(P)}$ is obtained from □ by relativizing all quantifers to the extension of P. Here P is a unary predicate symbol not appearing in □. To accommodate constant and function symbols, the antecedent needs to be strengthened to assert that P holds of the constants, and holds of values of the functions at arguments in the extension of P.

4. BASIC FORMAL THEORIES OF PREDICATION

In BTP (basic theory of predication), we have variables over W (W variables), in lower case; variables over general predicates on W, in upper case with superscript g; and variables over pure predicates on W, in upper case with superscript p. Also = on W, the binary function symbol < > on W, and constant symbol 0 (an object in W).

The W terms of BTP are:

1. Every W variable and 0 is a W term.
2. If s,t are W terms then <s,t> is a W term.

The atomic formulas of BTP are:

3. s = t, where s,t are W terms.
4. P^g(t),P^p(t), where P^g is a general predicate variable, P^p is a pure predicate variable, and t is a W term.

The formulas of BTP are:

5. atomic formulas are formulas.
6. closure under connectives.
7. closure under □,□.

Axioms and rules of BTP are:

A. Usual for L(BTP).
B. Pairing. <x,y> = <z,w> □ (x = z □ y = w).
C. Zero. <x,y> ≠ 0.
D. General Comprehension. (□P^g)(□x)(P^g(x) □ □), where □ in L(BTP) and P^g not free in □.
E. Pure Comprehension. (□P^p)(□x)(P^p(x) □ □), □ in L(BTP), P^p not free in □, no free W variables ≠ x in □, and no free general predicate variables in □.

Let BTPp is the axioms of BTP which have no general predicate variables; BTPg no pure predicate variables.

THEOREM 4.1. Theorems of BTP in L(BTPp) = theorems of BTPp; theorems of BTP in L(BTPg) = theorems of BTPg.

We can suitably develop arithmetic and finite sequences of W objects as W objects in either BTPp or BTPg. There is exactly one way to do this up to the appropriate kind of isomorphism. The p and g developments are demonstrably equivalent in BTP.

Induction with respect to all formulas will be provable in BTP.

We can also develop the satisfaction relation for any structure whose domain is W: In BTPp for pure structures, in BTPg for general structures.

Identity of indiscernibles:

$$\text{IIS)}\ (\square x,y)(x = y\ \square\ (\square P)(P^p(x)\ \square\ P^p(y))).$$

Singleton extensions principle:

$$\text{SEP)}\ (\square x)(\square P^p)(\square y)(P^p(y)\ \square\ y = x)$$

In BTPp, SEP implies IIS.

THEOREM 4.2. The implication IIS □ SEP is not provable in BTP.

5. COMPLETENESS OF PC, PC(=), PC(=,INF)

LPC = language of predicate calculus without equality, LPC(=) = language predicate calculus with equality. Let PC and PC(=) be the usual axioms and rules of inference.

Let PC(=,inf) be PC(=) plus:

$$(\square x_1,\ldots,x_n)(x_1 \neq \ldots \neq x_n),$$

where $n \geq 1$.

THEOREM 5.1. (Gödel) BTPg (BTPp) proves that a formula of LPC(=) is general (pure) domain satisfiable iff if it is consistent in PC(=).

We say that a sentence in PC(=) is Wg (Wp) satisfiable iff it holds in some general (pure) W structure.

We also say that a sentence in PC(=) is Ng (Np) satisfiable iff it holds in some general (pure) N structure.

THEOREM 5.2. (Gödel) BTPg (BTPp) proves that a sentence of LPC(=) is Ng (Np) satisfiable iff it is consistent in PC(=,inf).

The following result is proved by creating unlimited clones of any single element.

THEOREM 5.3. BTPg (BTPp) proves that a sentence of LPC is Wg (Wp) satisfiable iff it is consistent in PC.

Now we discuss these assertions:

Wg satisfiable = N satisfiable

Wp satisfiable = N satisfiable

Problematic. In one binary relation symbol R, look at:

R is a linear ordering.

This is Wg (Wp) satisfiable iff if there is a general (pure) linear ordering of W.

After failing to get your candidate through the hiring committe, you say

you can't rank order these candidates.

Maybe you could if there is a linear ordering of W.

THEOREM 5.4. The following are provably equivalent in BTPg (BTPp):

i) there is a general (pure) linear ordering of W;

ii) a sentence in PC(=) is Wg (Wp) satisfiable iff it is consistent in PC(=,inf).

Proof: We need only handle the p case. We argue in BTPp.

Suppose ii). Now

R is a linear ordering

is N satisfiable. Hence it is Wp satisfiable. I.e., there is a pure linear ordering of W.

It suffices to prove that i) implies ii). The forward direction of ii) is immediate.

Now suppose that there is a pure linear ordering of W. There is a pure dense linear ordering of W without endpoints by surrounding each point with a copy of the rationals.

Suppose □ is consistent in PC(=,inf). T has an explicitly arithmetical model M (with domain N) generated by explicitly arithmetic Skolem functions on an infinite set of explicitly arithmetic linearly ordered indiscernibles. We can assume that the indiscernibles have order type the rationals.

Introduce new constants c_x for each W object x. We define a structure whose domain D consists of the closed terms in these constants and the constant and function symbols of M. We will use the linear ordering of W, which linearly orders the subscripts of the new constants.

The truth value of any atomic formula will be determined by making any order preserving assignment of indiscernibles to the subscripts of the new constants appearing in the atomic formula, and setting it to be the truth value of the resulting statement in M. Because of indiscernibility, this is independent of the choice of order preserving assignment.

We then prove by induction that any formula in the language of M with assigned free variables (which are closed terms) holds* in this large structure iff it holds in M after the closed terms are changed to corresponding elements of D. Part of the in-duction hypothesis is that the truth value in M so obtained does not depend on the choice of order preserving map.

Note the asterisk at the sticky point. The interpretation of = is not equality, but the equivalence relation between these closed terms, according to whether two closed terms have equal values in M when the subscripts of the constants appearing are mapped by an order preserving map into the original indiscernibles.

Normally, this is remedied by calling the large structure a weak model of the sentences true in M, and then factoring by the equivalence relation.

The problem here is that the ensemble of equivalence classes is not like an extension of a pure predicate - the type is too high.

We need to find a pure function H from D into W such that two terms are equivalent iff their values under H are equal. The size of the image of H must be the same as W because the new constants all lie in different equivalence classes.

We can then factor D by the equivalence relation using values of H to obtain a pure structure whose domain is large; i.e., such that there is a one-one pure map from W into the domain. But then we use the Shroeder Bernstein theorem in this context, which can be proved in its pure form in BTPp. QED

THEOREM 5.5. BTP neither proves nor refutes any equality between Wg satisfiability, Wp satisfiability, and consistency in PC(=,inf).

6. PRINCIPLE OF SYMMETRIC ARGUMENTS

A sentence in LPC(=) is said to be existential (universal) iff it begins with a block of zero or more existential (universal) quantifiers followed by a quantifier free formula.

THEROEM 6.1. It is provable in BTPg (BTPp) that every existential sentence is Wg (Wp) satisfiable iff it is consistent in PC(=).

It is sometimes convenient to consider the Wg (Wp) valid formulas. A formula is Wg (Wp) valid iff its universal closure is Wg (Wp) valid iff its negation is not Wg (Wp) satisfiable.

The most basic sentence of PC(=) whose Wg (Wp) validity is in question is

$$(\exists x)(\exists y)(x \neq y \wedge (R(x,y) \leftrightarrow R(y,x))).$$

The general (pure) principle of binary symmetric arguments asserts that this sentence is Wg (Wp) valid.

The general (pure) principle of symmetric arguments asserts that for all $k \geq 1$, every general (pure) predicate holds or fails of all permutations of some k-tuple of distinct objects.

Note that this is trivial for $k = 1$. Also, if it holds for $k \geq 2$ then it holds for $k-1$.

THEOREM 6.2. The general and pure principles of symmetric arguments are neither provable nor refutable from BTP. The general principle does not follow from the pure principle in BTP.

We say that a general (pure) predicate P is finite iff there is a finite sequence x such that

(□y)(P(y) □ y is a term in x).

We say that a general (pure) predicate P is infinite iff it is not finite.

THEOREM 6.3. BTP proves that a general predicate is finite iff its extension is in general one-one correspondence with a proper initial segment of N. BTP does not prove all finite pure predicates are in pure one-one correspondence with a proper initial segment of N.

A minimally infinite general predicate is an infinite general predicate P such that no general predicate splits P. I.e.,for any general predicate Q, either

i) there is a finite sequence x such that (□y)((P(y) □ Q(y)) □ y is a term in x); or
ii) there is a finite sequence x such that (□y)((P(y) □ □Q(y)) □ y is a term in x).

THEOREM 6.4. BTPg proves: if □ minimally infinite general predicate then general principle of symmetric arguments. Converse not provable in BTP.

THEOREM 6.5. BTP does not prove or refute □ a minimally infinite general predicate.

An absolute POI (predicate of indiscernibles) is a pure predicate P where any pure predicate holds or fails of any two equal length finite sequences of distinct objects from the extension of P.

THEOREM 6.6. BTP does not prove or refute the existence of an infinite absolute POI.

7. SATISFIABILITY OF UNIVERSAL SENTENCES

SYM(=) is PC(=) plus: Let $k \geq 1$ and φ be a formula in LPC(=) with at most the free variables $x_1,\ldots,x_k$. Take

$$(\forall x_1 \neq \ldots \neq x_k)(\text{conjunction of } (\varphi(x_1,\ldots,x_k) \leftrightarrow \varphi(x_{\sigma 1},\ldots,x_{\sigma k}))),$$

conjunction ranging over all permutations σ of $\{1,\ldots,k\}$.

THEOREM 7.3. BTPp proves that every universal sentence consistent in SYM(=) is Wp satisfiable.

Thus for universal sentences, consistency in SYM(=) implies Wp satisfiability implies consistency in PC(=,inf).

THEOREM 7.4. The following are provably equivalent in BTPg (BTPp).
i) the general (pure) principle of symmetric arguments;
ii) every universal sentence is Wg (Wp) satisfiable iff it is consistent in SYM(=).

Thus if you accept the prin-ciple of symmetric arguments, then you have completely determined the universal sentences that are W satisfiable.

8. NECESSARY SATISFIABILITY OF UNIVERSAL SENTENCES

Another approach to Wg (Wp) satisfiability is to determine which sentences are possibly or necessarily Wg (Wp) satisfiable relative to BTP.

We say that a sentence φ is possibly (necessarily) Wg satisfiable over BTP iff "φ is Wg satisfi-able" is consistent (provable) in BTP. The same for Wp.

THEOREM 8.1. A sentence is possibly Wg (Wp) satisfiable over BTP if and only if it is satisfiable.

A relational sentence is one which has no constant or function symbols.

THEOREM 8.2. A universal relational sentence is necessarily Wg (Wp) satisfiable over BTP iff it is consistent in SYM(=).

THEOREM 8.3. A universal sentence is necessarily Wg (Wp) satisfiable over BTP iff the assertion that it is consistent in SYM(=) is provable in the formal system of second order arithmetic. A universal sentence is neces-sarily Wg (Wp) satisfiable over BTP + the true arithmetic sentences iff it is consistent in SYM(=).

9. AN ALTERNATIVE AXIOMATIZATION

Let $d_1,d_2,...$ be new constants. Let □ be a formula in LPC(=) with at most the free variables $x_1,...,x_k$ and □ be a per-mutation of (1,...,k). Take

$$□(d_1,...,d_k) \; □ \; □(d_{□1},...,d_{□k}).$$

We write this as SYMC(=), where C means "constants."

THEOREM 9.1. SYM(=) and SYMC(=) prove the same formulas in LPC(=).

So we can use the purely universal theory SYMC(=) instead of SYM(=).

10. ALTERNATIVE DETERMINATIONS

We may deny the general (pure) principle of symmetric argu-ments. Then the Wg (Wp) satisfiable universal sentences will be properly greater than those provable in SYM(=). Can we adopt some of the general (pure) principle of symmetric arguments without adopting all of it?

THEOREM 10.1. Let $k \geq 2$. It is consistent with BTP that the general principle of k-ary symmetric arguments holds but not the general principle of k+1-ary symmetric arguments.

This leads to a determination of the Wg satisfiable universal sentences that is intermediate between consistency with SYM(=) and satisfiability. There are a number of further questions here that we have not investigated.

11. SATISFIABILITY OF □□ SENTENCES

We characterize the Wg (Wp) satisfiable □□ sentences by allowing parameters in SYM(=).

SYM'(=) is PC(=) plus: Let $k \geq 1$ and □ be a formula in LPC(=). Take

$$(□x_1 \neq ... \neq x_k)(\text{conjunction of}$$

$$(□(x_1,...,x_k) □ □(x_{□1},...,x_{□k}))),$$

conjunction ranging over all permutations □ of {1,...,k}. Free variables are interpreted universally.

THEOREM 11.1. A universal sentence is consistent in SYM'(=) if and only if it is consistent in SYM(=).

THEOREM 11.2. The following are provable in BTPp. Every □□ sentence consistent in SYM'(=) is Wp satisfiable.

THEOREM 11.3. The following are provably equivalent in BTPg (BTPp).
i) the general (pure) principle of symmetric arguments;
ii) every □□ sentence is Wg (Wp) satisfiable iff it is consistent in SYM'(=).

12. ARBITRARY FORMULAS - MODEL THEORETIC DETERMINATION

Here we give a determination of the Wg and Wp validity of arbitrary formulas in LPC(=) in model theoretic terms. In particular, the sentences determined to be Wg (Wp) satisfiable will be co r.e.

Let M be a structure, E □ dom(M). We say E is a set of symmetric generators of M iff every element of dom(M) is a term of elements of E, and every permutation of E extends to a permutation of M.

Let SYMGEN(=) be the sentences that hold in some structure with an infinite set of sym-metric generators. The type of the structure may have to be include more symbols than those appearing in □.

THEOREM 12.1. SYMGEN(=) is complete co r.e. An □□ sentence lies in SYMGEN(=) iff it is consistent in SYM'(=). A universal sentence lies in SYMGEN(=) iff it is consistent in SYM(=).

THEOREM 12.2. The following is provable in BTPg (BTPp). Every sentence in SYMGEN(=) is Wg (Wp) satisfiable.

Let P be a general k+1-ary predicate.

We say that F is a choice function for P if and only if

i) F is a function of the form $F:A^k \to A$, where A is the extension of a general predicate;
ii) for all $x \in A^k$, if $(\exists y)$(P(<x,y>) then P(<x,F(x)>).

Principle *) asserts that every general multivariate predicate has a choice function with an infinite set of symmetric generators.

THEOREM 12.3. BTP does not prove or refute principle *). The following are provably equivalent in BTPg.
i) principle *);
ii) any sentence that is Wg satisfiable lies in SYMGEN(=);
iii) a sentence is Wg satisfiable iff it lies in SYMGEN(=).

We think that *) goes against the idea behind pure predication. So we give a weakened form of *) involving a finite conclusion.

Let P be a general (pure) k+1-ary predicate. We say that F is a finite choice function for P if and only if

i) F is a function with domain A^k, where A is given by a finite sequence;
ii) for all $x \in A^k$, if $(\exists y)$(P(<x,y>) then P(<x,F(x)>).

We say that E is n-symmetric in F iff

i) every permutation of E extends to an automorphism of F;
ii) every term involving F and at most n occurrences of elements of E is defined.

Principle **g) says: for all n,m every general multivariate predicate has a choice func-tion with a set of n-symmetric generators of cardinality m. **p) is the same with general replaced by pure.

THEOREM 12.4. BTP does not prove or refute principle **g) or **p). BTPg (BTPp) proves the equivalence of:
i) principle **) for general (pure) predicates;
ii) any sentence that is Wg (Wp) satisfiable lies in SYMGEN(=);
iii) a sentence is Wg (Wp) satisfiable if and only if it lies in SYMGEN(=).

13. NECESSARY SATISFIABILITY OF SENTENCES

THEOREM 13.1. A sentence is necessarily Wg (Wp) satisfiable over BTP iff the assertion that it has a countable model with an infinite set of symmetric generators is provable in the formal system of second order arithmetic.

THEOREM 13.2. A sentence is necessarily Wg (Wp) satisfi-able over BTP + the true arithmetic sentences iff it has a model with an infinite set of symmetric generators.

14. CLASS THEORETIC INTERPRETATION OF PREDICATION ON W

The theory of classes provides appropriate models of predication on W that are familiar.

W is interpreted as the class V of all sets. 0 is interpreted as the empty set. Pairing is interpreted as a standard pairing function in set theory. Equality is inter-preted as equality in set theory. Predicates on W are interpreted as classes.

We have a number of options. We can allow/disallow urelements, allow/disallow the axiom of choice for sets. We then interpret general predicates on W as arbitrary classes.

The distinction between pure and general predicates - classes here - is best made axiomatically by disallowing set parameters in the comprehension axiom scheme for pure classes.