ADVENTURES IN INCOMPLETENESS

by
Harvey M. Friedman
Distinguished University Professor
Mathematics, Philosophy, Computer Science Emeritus
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Thanks Tom for the unexpected invitation! I'm on a family based trip right t now, but thought I would see if I could run my thoughts about how f.o.m. = foundations of mathematics can productively interact with many diverse areas of Philosophy, including those areas for which the researchers thought that they were protected from such invasions.

My idea is that f.o.m. can be viewed as having grown out of considerations from Philosophy of Mathematics, Philosophy of language, Philosophical Logic, metaphysics, and epistemology, into an autonomous philosophically informed and motivated mathematically oriented rich and deep area of great power of depth. I view f.o.m. as a singular success story for Philosophical Thinking.

I don't have the time to talk here about f.o.m. in general, but I do want to make some small points so that you have a better idea of where I am coming from. Am I a mathematician or a philosopher? Well, depending on just what you mean, I am arguably both and arguably neither.

I like to draw a distinction between f.o.m. and mathematical logic. The latter is a focused area of mathematics that investigates the great fundamental f.o.m. systems and structures for their own sake or for the sake of interactions with other areas of mathematics. There have also been notable interactions with computer science.

On the other hand, f.o.m. is primarily philosophically motivated, and shines most brightly in the general philosophical realm of "foundations of subjects". I of course look to Kurt Gödel as the great seminal figure in f.o.m. at least post 1900. He represented that exquisite delicate balance between the philosophical and the mathematical that is so required and so effective for f.o.m.

It has been highly challenging to seriously build on the great legacies of Kurt Gödel. The most famous, but by no means the only, of these legacies, is Incompleteness. This is where I have put my greatest efforts.

Incompleteness, in a general sense, started long before Gödel, although not commonly referred to as such. The arguably most shocking revelation in f.o.m., given the context in which it appeared, can be naturally formulated as the first monumental event in Incompleteness.

Consider the well known standard ordered field axioms, OFA. This is based on 0,1,+,-,x,1/,<. These familiar axioms are at the heart of the elementary school math curriculum. Now consider

There exists b such that b x b = 1+1.

It was more or less discovered 2000 years ago that, in modern terms, the above statement is neither provable nor refutable in OFA. I.e., the above is independent of OFA. In modern terms, we have a model of OFA in which this is true, and a model of OFA in which this is false. It is true in the real numbers, and false in the rational numbers.

Fast forward around 2000 years, and we know how to fix this Incompleteness. There are two well known ways.

1. Add: If b > 0 then there exists c such that b = c x c. Every polynomial of odd degree in one variable has a root.
2. Add: The least upper bound principle for all first order formulas.

Both 1,2 use infinitely many axioms and this is unavoidable. They are both equivalent and this is not at all obvious. They entirely stamp out all incompleteness: the resulting systems prove or refute all statements in its LANGUAGE. And note that 1,2 are satisfactory in the sense that instances are easily algorithmically recognized.

There are similar developments, more involved, connected with elementary geometry instead of this elementary algebra, with a particularly famous example of the parallel postulate in Euclidean geometry. The Incompletenesses are again fixable. In many cases in geometry, however, there is a much stronger kind of Incompleteness - and also fixable. This is the so called second order Incompleteness, and relationships between first order and second order Incompleteness is itself worthy of at least several talks here. We will confine ourselves to first order incompleteness.

Now let us turn to the discrete ordered ring axioms, DORA. This is very much like OFA except that we only think of integers - no reciprocal or division. This is also an elementary school system, with 0,1,+,-,x,<. But instead of anything about reciprocal/division, we add

Nothing is strictly between 0 and 1.

Now consider this very basic

For all b there exists c such that c+c = b or c+c = b+1.

This statement is independent of DORA. Let's use idea #2 for trying to fix this Incompleteness:

2*. Add: The least upper bound principle for all first order formulas.

GOEDEL. DORA + 2* still has Incompleteness. In fact, there is no way to add further axioms to appropriately fix this Incompleteness.

DORA + 2* is essentially a rewrite of what is normally called PA = Peano Arithmetic.

PA may well be strong enough to prove or refute all individual mathematical statements that have, as of January 24, 2017, been published in accepted mathematical venues by mathematicians operating as mathematicians according to normal mathematical culture, as opposed to acting as f.o.m. investigators (like me). E.g., it is widely believed that FLT = Fermat's Last Theorem is provable in PA, although this has not yet been firmly established.

Nevertheless, this leaves open the possibility that f.o.m. investigators may be able to discover a statement in the language of PA that is independent of PA, that can be argued to be fully compatible with normal mathematical culture, perhaps enough so that the statement can be argued to be inevitable over the realistically far out future of mathematics.

In fact, we now have a growing body of ever more convincing examples of this - what I call CONCRETE MATHEMATICAL INCOMPLETENESS.

The early examples of such Concrete Mathematical Incompleteness at the PA level are Goodstein's Theorem (1944, 1982), Paris/Harrington Theorem (1977), Hydra Game (1982). There are some later examples at the PA level that are arguably more aligned with ordinary mathematical culture, discussed in PA Incompleteness on FOM at http://www.cs.nyu.edu/pipermail/fom/2016-September/020083.html. At this point, my favorite is

DEFINITION. For $x,y \in N^k$, x <adj y means x,y are each strictly increasing and $(x_2,\ldots,x_k) = (y_1,\ldots,y_{k-1})$. Note that this proscribes a single order type for the 2k-tuple (x,y). x ≤c y means that each $x_i \leq y_i$.

ADJACENT LIFTING. Every $f:N^k \to N^k$ has some x <adj y with f(x) ≤c f(y).

RECURSIVE ADJACENT LIFTING. Every recursive $f:N^k$ into N^k has some x <adj y with f(x) ≤c f(y).

ELEMENTARY RECURSIVE ADJACENT LIFTING. Every elementary recursive $f:N^k \to N^k$ has some x <adj y with f(x) ≤c f(y).

POLYNOMIAL ADJACENT LIFTING. Every surjective polynomial $P:N^k \to N^k$ has some x ≤c y with P(x) <adj P(y).

The first is equivalent to ϵ_0 is well ordered, the second to 2-Con(PA), the third to 1-Con(PA), the fourth to 2-Con(PA). These four results use base theory RCA0 for the first and EFA for the remaining. For the first three results, see http://cage.ugent.be/~pelupessy/ARPH.pdf Reference [7] there has the wrong URL. It should be http://u.osu.edu/friedman.8/files/2014/01/PA-incomp082910-2lgh5wm.pdf

For the fourth result, see PA Incompleteness/2, FOM email list, February, 2017, http://www.cs.nyu.edu/pipermail/fom/ This improves on http://www.cs.nyu.edu/pipermail/fom/2013August/017469.html

There is a 250 page Introduction to a book draft at https://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/ covering the state of Concrete Mathematical Incompleteness through BRT = Boolean Relation Theory.

This history includes a variety of natural concrete mathematical examples of Incompleteness at various levels from below PA to mostly around "uncountably many iterations of the power set operation", a very substantial fragment of ZFC. It also includes BRT, and has not been updated to include EMULATION THEORY - its fleshing out is my highest priority for 2017. A full picture of Concrete Mathematical Incompleteness can only be properly covered in a series of talks.

In every single existing case of Concrete Mathematical Incompleteness, we have the following common situation. A natural Concrete Mathematical statement A is shown, over an appropriately very weak system, to be provably equivalent to the consistency of an unexpectedly strong system T - or some standard variant of consistency (such as 1-consistency).

Assuming T is "OK", this establishes the independence of A from T. For if A were refutable from T then T would not be OK. And if A were provable from T then T would prove its consistency, and then by Gödel's Second Incompleteness Theorem, T would be inconsistent - definitely not OK.

We now jump to EMULATION THEORY. We need to shorten the full story some and so for good reason we are going to work in $Q[0,1]^k$. Here Q[0,1] is the closed unit internal in the rationals Q.

DEFINITION 1.1. We say that $S \subseteq Q[0,1]^2$ is drop equivalent at (x,y),(x',y) if and only if for all z < y, (x,z) in S iff (x',z) in S.

Let's draw a picture for this crucial drop equivalence.

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 _______________
|               |
|    A   B      |
|    |   |      |
|    |   |      |
|____|___|______|
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This rectangle is $Q[0,1]^2$ with the points A = (x,y), B = (x',y). We have a set $S \subseteq Q[0,1]^2$ in the background. As we drop from A and from B, we want each point below A to lie in S if and only if the corresponding point below B lies in S.

THEOREM 1.1. There exists $S \subseteq Q[0,1]^2$ where drop equivalence holds only trivially. I.e., S is drop equivalent at $(x,y),(x',y)$ if and only if $x = x' \vee y = 0$.

We can repair Theorem 1.1 at some cost.

THEOREM 1.2. Every $S \subseteq Q[0,1]^2$ is drop equivalent at some $(x,y),(x',y)$, $x \neq x' \wedge y > 0$, if we replace Q[0,1] by some other dense linear ordering with endpoints 0,1. These replacements can be of any uncountable cardinality but not countable.

So far we are not threatening ZFC. However, look at this:

THEOREM 1.3. Every $S \subseteq Q[0,1]^2$ is drop equivalent at some $(x,x),(x',x)$, $0 < x < x'$, provided we replace Q[0,1] by some gigantic dense linear ordering with endpoints 0,1. The size required here is far beyond anything that can be proved to exist in ZFC.

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 _______________
|               |
|          A  B |
|          |  | |
|          |  | |
|__________|__|_|
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Here A is on the diagonal. Don't get excited yet! This is an example of Mathematical Incompleteness that is closely related to well known developments in large cardinal theory. The statement is intensely set theoretic, and we already have a range of Mathematical Incompleteness in the highly set theoretic realm.

Emulation Theory gets to the essence of Theorem 1.3 while staying in Q[0,1]! Obviously we are going to have to add some very potent special sauce to pull this off, and the idea is to make this special sauce delicious. So delicious that ultimately mathematicians will want to use this special sauce across the whole realm of Concrete Mathematics! OK, we are still preparing the sauce, and we are looking out to the future.

The large cardinals involved in Theorem 1.3 are treated in

H. Friedman, Subtle Cardinals and Linear Orderings, Annals of Pure and Applied Logic, Volume 107, Issues 1-3, 15 January 2001, Pages 1-34.
https://u.osu.edu/friedman.8/files/2014/01/subtlecardinals-1tod0i8.pdf

PROTOTYPE 1. For subsets of $Q[0,1]^2$, some MAXIMAL EMULATION is drop equivalent at some $(x,x),(x',x)$, $0 < x < x'$.

Thus we don't use any old subset of $Q[0,1]^2$, but rather some sort of cousin.

Maximal Emulations, yet to be defined, will be allowed to move rationals around in order preserving ways. We will be using only the order on Q[0,1], and NOTHING more. This means that we have a very nice SIMPLIFICATION here. We can say what x,x' are IN ADVANCE. We will use the friendly numbers 1/2,1. It is by no means automatic that we can use the endpoint 1 for this purpose, but it turns out that we can with good effect. So we have the following SIMPLIFIED prototype:

PROTOTYPE 2. For subsets of $Q[0,1]^2$, some MAXIMAL EMULATION is drop equivalent at (1/2,1/2),(1,1/2).

The above is the Lead Statement in Emulation Theory for dimension 2! A Maximal Emulation is an emulation which is not a proper subset of any emulation.

Of course, I haven't yet told you what an emulation is. Emulation is given by a fundamental equivalence relation between subsets of $Q[0,1]^2$ that only involves the ordering on Q[0,1].

DEFINITION 1.2. $x,y \in Q^k$ are order equivalent if and only if their coordinates have the same relative order. I.e., for all $1 \leq i,j \leq k$, $x_i < x_j$ iff $y_i < y_j$. S is a 1-emulation of E $\subseteq Q[0,1]^2$ if and only if E,S have the same elements up to order equivalence.

THEOREM 1.4. There are exactly 8 equivalence classes of subsets of $Q[0,1]^2$ under 1-emulation.

Proof: The number of equivalence class of elements of $Q[0,1]^2$ under order equivalence is 3. So the count is $2^3 = 8$. QED

MED/1. For subsets of $Q[0,1]^2$, some maximal 1-emulation is drop equivalent at (1/2,1/2),(1,1/2).

MED is read "maximal emulation drop" and /1 indicates that it is the first in the MED series.

But MED/1 is actually very easy to prove. This is because maximal 1-emulations are so simple. Every maximal 1-emulation (regardless of what it is 1-emulating) is merely an equivalence class under the equivalence relation of order equivalence on $Q[0,1]^2$. And it is an easy exercise that every such equivalence class is automatically drop equivalent at (1/2,1/2),(1,1/2), and in fact at any (p,p),(p',p), $0 < p < p'$.

Also every subset of $Q[0,1]^2$ has a maximum 1-emulation. So we get this extremely strong form of MED/1:

MED/2. For subsets of $Q[0,1]^2$, the maximum 1-emulation is drop equivalent at every (p,p),(p',p), $0 < p < p'$.

But that is just 1-emulation. The official definition of Emulation, also written as 2-emulation, is as follows.

DEFINITION 1.3. S is an emulation of E $\subseteq Q[0,1]^2$ if and only if E,S have the same pairs of elements (pairs of pairs!) up to order equivalence of 4-tuples. More generally, S is an r-emulation of E $\subseteq Q[0,1]^2$ if and only if E,S have the same r-tuples of elements up to order equivalence of 2r-tuples.

The idea behind emulation is that emulations have the same pairs (of pairs) up to order equivalence.

An exact count on the number of equivalence classes of subsets of $Q[0,1]^2$ under emulation is not straightforward. Here we will only give a rough estimate, although we believe that an exact count is definitely achievable as a nice piece of elementary finite combinatorics.

Exact counts for small k of the number of equivalence classes of elements of $Q[0,1]^k$ under order equivalence - a much simpler and prior problem - are listed in https://u.osu.edu/friedman.8/foundational-adventures/downloadable-manuscripts/ #76 page 7 (lifted from another source).

THEOREM 1.5. The number of equivalence classes of subsets of $Q[0,1]^2$ under Emulation is at least 2^{30} and at most 2^{39}.

MED/3. For subsets of $Q[0,1]^2$, some maximal emulation is drop equivalent at (1/2,1/2),(1,1/2).

MED/4. For subsets of $Q[0,1]^2$, some maximal r-emulation is drop equivalent at (1/2,1/2),(1,1/2).

What is the status of MED/3 and MED/4? Are these provable in ZFC?

Don't get exited yet. I know how to prove MED/4 using the existence of an uncountable set. My proof does not go through in countable set theory, which is essentially ZFC\P. I suspect that MED/3 can be proved in ZFC\P but MED/4 cannot.

Now I am going to put the Fasten Seat Belt Sign On. We are going to lift off to THREE DIMENSIONS!

Emulations (2-emulations) in 3 dimensions involves 6 tuples just like emulations (2-emulations) in 2 dimensions involves 4 tuples. Instead of using (1/2,1/2),(1,1/2), we use (1/2,1/3,1/3),(1,1/2,1/3).

We find it more readable to switch these two triples.

MED/5. For sunsets of $Q[0,1]^3$, some maximal emulation is drop equivalent at (1,1/2,1/3),(1/2,1/3,1/3).

Thus we are requiring that for all $p < 1/3$, $(1,1/2,p) \in S \leftrightarrow (1/2,1/3,p) \in S$. Thus we are dropping vertically from the two points (1,1/2,1/3),(1/2,1/3,1/3) in the cube $Q[0,1]^3$ down to the base $z = 0$.

The only proof that we have of MED/5 uses the large cardinals mentioned before in connection with the set theoretic Theorem 1.3 above. We judge about a 50//50 chance that MED/5 can be proved in ZFC. However, consider

MED/6. For sunsets of $Q[0,1]^3$, some maximal 8-emulation is drop equivalent at (1,1/2,1/3),(1/2,1/3,1/3).

Here we expect that far more than ZFC is required to prove MED/6. The only definite claim we are making at this point is that

MED/7. For sunsets of $Q[0,1]^k$, some maximal r-emulation is drop equivalent at (1,1/2,...,1/k),(1/2,...,1/k,1/k).

cannot be proved in ZFC, and in fact is provably equivalent, over WKL_0, to the consistency of the system SRP. This is ZFC + $\{$there exists a subtle cardinal of order k$\}_k$. We hope that the situation clarifies this calendar year.

Note that these statements still quantify over countably infinite objects. So we can naturally demand more and much more Concreteness. Emulation Theory addresses this in three different ways.

1. The Implicit way. The logical form of, e.g., MED/7 is such that it is an easy undergraduate math logic exercise to reformulate it as asserting that an effectively given list of sentences in first order predicate calculus with equality each have a countable model. Then by Gödel's famous Completeness (not Incompleteness) Theorem, MED/7 is equivalent to an explicitly finite statement, with the infinite objects removed. Of course, we are still quantifying over infinitely many finite objects. The explicitly finite statement obtained in this way is in Π^0_1 form.

2. A consequence of 1 is the hallmark property of Π^0_1 statements such as FLT. We know, a priori, that if the statement is false then it can in principle be verified to be false.

3. However, the Π^0_1 forms using 1 lose their purely mathematical character. We also have a way of dissecting maximal emulations via finite approximations, to directly obtain explicitly Π^0_1 forms which, at least arguably,

maintain the purely mathematical character.

Emulation Theory also touches practically the strongest of the large cardinal hypotheses that have been proposed. At this point, some extra ingredients are needed for this, which are steadily becoming more compatible with ordinary concrete mathematical culture.