APPLICATIONS OF LARGE CARDINALS TO BOREL FUNCTIONS

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NOTE: This is work in progress. No proofs are presented. Results are still being checked.

Let $R$ be the set of all real numbers, and let $CS(R)$ be the space of all nonempty countable subsets of $R$.

The space $CS(R)$ has a unique “Borel structure” in the following sense. Note that there is a natural mapping from $R^\omega$ onto $CS(R)$; namely, taking ranges. We can combine this with any Borel bijection from $R$ onto $R^\omega$ in order to get a “preferred” surjection $F:R \to CS(R)$.

In what sense is this preferred? Consider the following property $*$ on $F:R \to CS(R)$:

i) $F$ is onto;
ii) $\{(x,y_1,y_2,...): F(x) = F(y_1)U F(y_2)U ...\}$ is a Borel measurable subset of $R^\omega$.

By way of background, we have the following:

THEOREM 1. Let $F,G:R \to CS(R)$ have property $*$. Then $G$ is the result of composing $F$ with a Borel permutation of $R$.

In light of Theorem 1, we fix a preferred $\varphi:R \to CS(R)$.

There are two reasonable ways to define the Borel functions $F$ from $CS(R)$ into $CS(R)$.

1. There exists Borel $G:R \to R$ such that $F(\varphi(x)) = \varphi(G(x))$.
2. $\{(x,y): F(\varphi(x)) = \varphi(y)\}$ is Borel measurable subset of $R^2$.

THEOREM 2. Both of these definitions of Borel $F:CS(R) \to CS(R)$ are equivalent.
The following basic result indicates the likelihood of a substantial theory of the structure of Borel functions on \( \text{CS}(R) \).

**THEOREM 3.** Let \( F: \text{CS}(R) \to \text{CS}(R) \) be Borel. Then there exists \( A \) such that \( F(A) \subseteq A \).

We proved this around 1977. We actually showed that this can be proved in third order arithmetic but not in second order arithmetic. See [Fr].

We now want to talk about a new theorem of this rough form (Borel diagonalization) which is independent of ZFC.

Let \( X \) be an uncountable complete separable metric space. Then we can discuss Borel functions on \( \text{CS}(X) \) in the same manner.

More generally, let \( Y \) be an uncountable Borel measurable subset of \( X \). We can also consider \( \text{CS}(Y) \). Using any Borel measurable bijection between \( X \) and \( Y \), we can define the Borel functions on \( \text{CS}(Y) \).

We say that \( x, y \in R^\infty \) are finitely equivalent if and only if \( y \) is obtained from \( x \) by a permutation of the indices that leaves all but finitely many indices fixed.

We say that \( A \subseteq R^\infty \) is finitely invariant if and only if \( x \in A \) and \( E(x, y) \) implies \( y \in A \). We write \( \text{FICS}(R^\infty) \) for the space of all nonempty finitely invariant countable subsets of \( R^\infty \). This is obviously an uncountable Borel subset of \( \text{CS}(R^\infty) \), and therefore we can consider Borel functions on \( \text{FICS}(R^\infty) \) in the usual way.

Let \( x, y \in R^\infty \). We say that \( x \) is a subsequence of \( y \) if and only if there is a strictly increasing function \( f:N \to N \) such that each \( x_i = y_{f(i)} \).

Here is a warmup exercise.

**THEOREM 4.** Let \( G:\text{FICS}(R^\infty) \to \text{FICS}(R^\infty) \) be Borel. Then there exists \( A \) such that every element of \( G(A) \) is a subsequence of an element of \( A \).

Theorem 4 has a proof that is closely related to Theorem 3, and so is provable in third order arithmetic but not in second order arithmetic.
We say that $A \in \text{FICS}(\mathbb{R}^\omega)$ is a chain if and only if for all $x, y \in A$, $x$ is a subsequence of $y$ or $y$ is a subsequence of $x$.

**THEOREM 5.** Let $G: \text{FICS}(\mathbb{R}^\omega) \to \text{FICS}(\mathbb{R}^\omega)$ be Borel. Then there exists a chain $A$ such that every element of $G(A)$ is a subsequence of an element of $A$.

It is necessary and sufficient to use infinitely many uncountable cardinals to prove Theorem 5. Theorem 5 cannot be proved in Zermelo set theory, but can be proved in $\text{ZF}\setminus\text{P} + V(\omega+\omega)$ exists.

Now for the big stuff.

**THEOREM 6.** Let $G: \text{FICS}(\mathbb{R}^\omega) \to \text{FICS}(\mathbb{R}^\omega)$ be Borel. Then there exists $A$ such that all elements of values of $G$ at subsets of $A$ are subsequences of elements of $A$.

Theorem 6 can be proved from a measurable cardinal, yet not with "every subset of $\mathbb{N}$ has a sharp." Presumably, $\text{ZFC} + \text{Ramsey cardinal}$ should also not suffice.

Again, in light of Theorems 4, 5, 6, there should be a substantial structure theory for the Borel functions on the space $\text{FICS}(\mathbb{R}^\omega)$.

We are working on getting a clean extension of Theorem 6 that would require many measurable cardinals to prove.